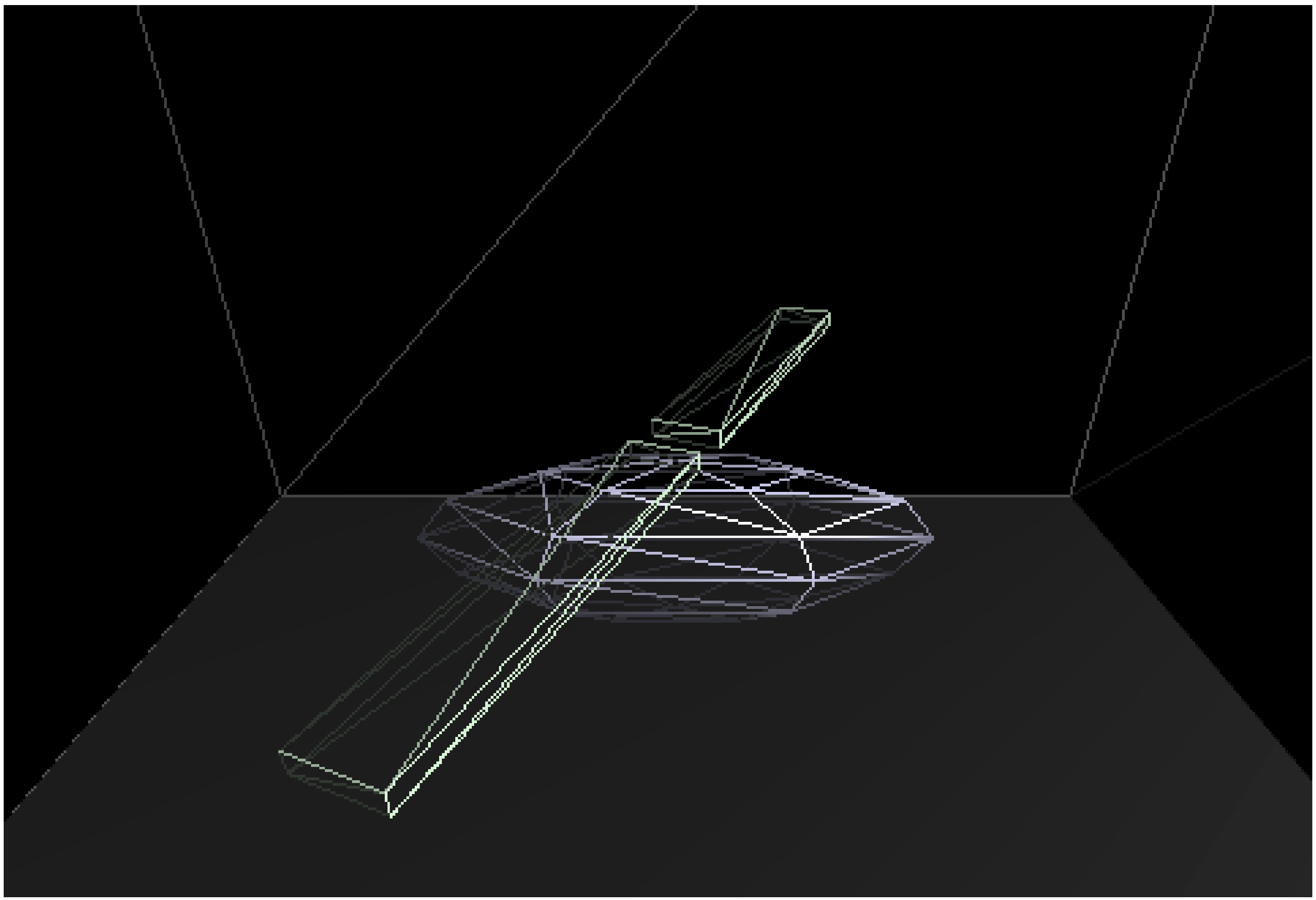
Computer Graphics

MTAT.03.015

Raimond Tunnel

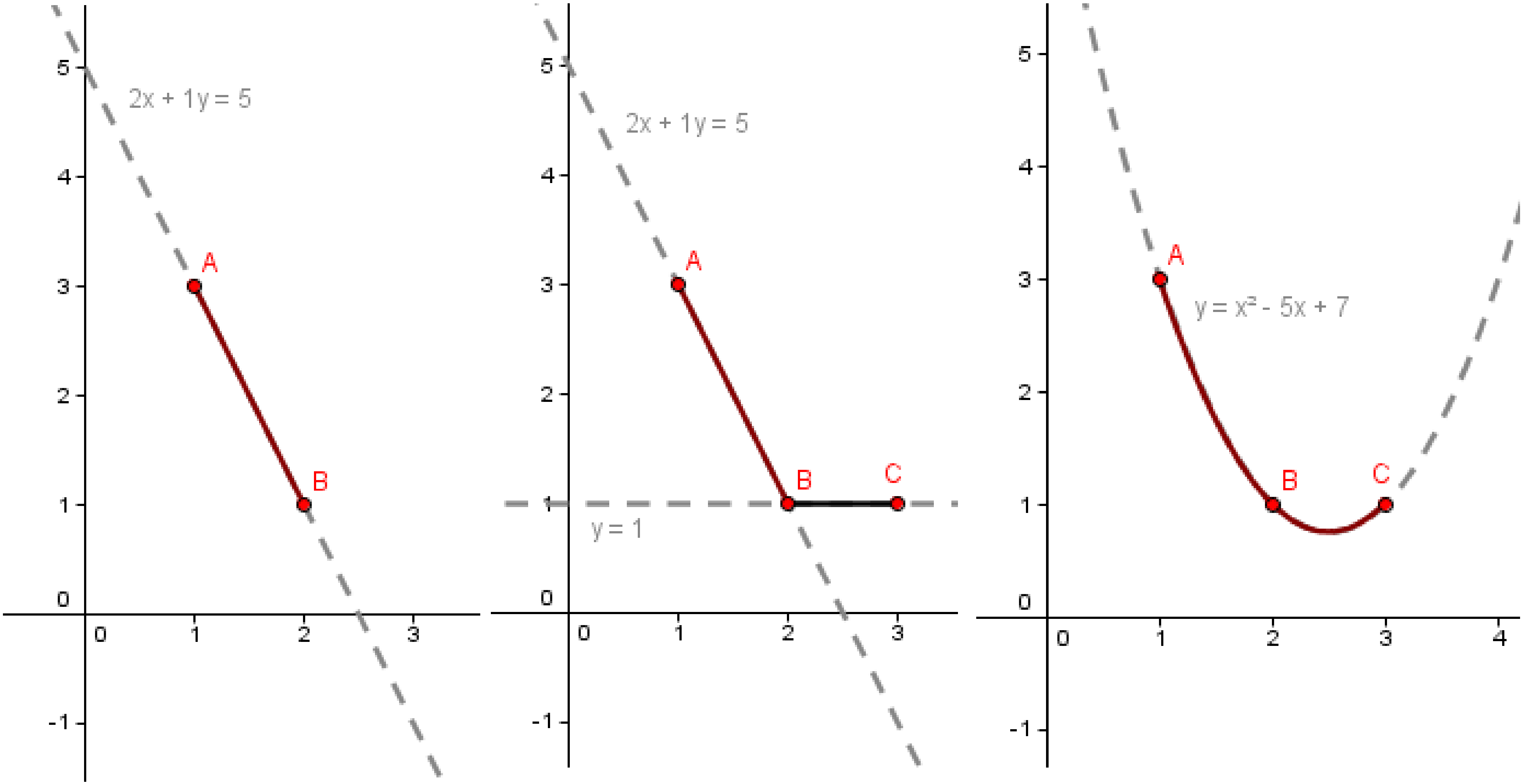
The Road So Far...



Curves

- Line interpolates between 2 points.
- Mathematically there higher polynomials to interpolate between more points
- How many points you need, to construct a *n*-th degree polynomial through it?

Curves



Curves

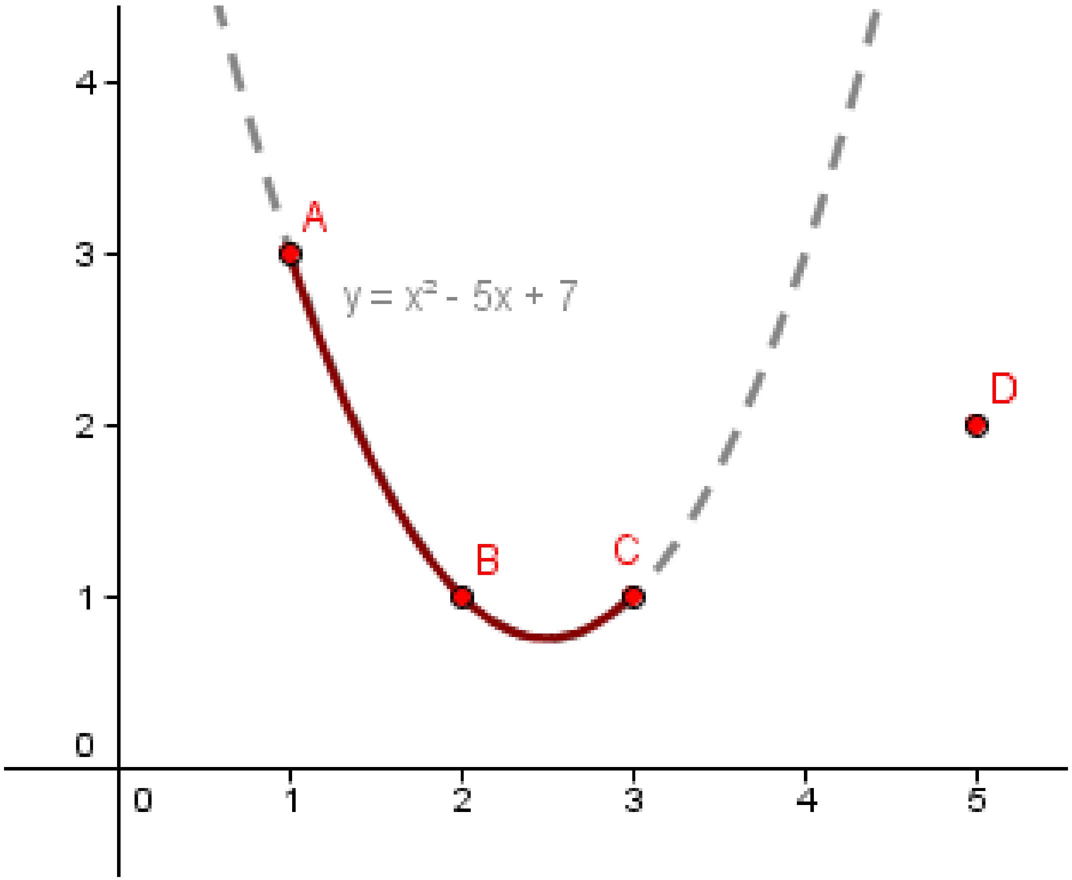
- How to construct a parabola through 3 points?

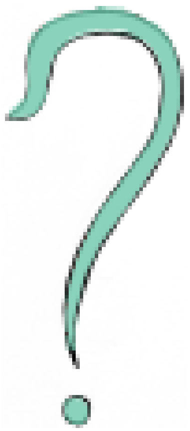
$$f(x) = a_2 \cdot x^2 + a_1 \cdot x + a_0$$

- We are looking for: $a_{2,}\ \ a_{1,}\ \ a_0$.
- We know: $f(1) = 3,\ \ f(2) = 1,\ \ f(3) = 1$
- 3 unknowns, 3 constraints, we can solve it.
- http://www.wolframalpha.com/input/?i=a*1+%2B+b*1+%2B+c+%3D+3%2C+a*4+%2B+b*2+%2B+c+%3D+1%2C+a*9+%2B+b*3+%2B+c+%3D+1

Curves

- What choices we have with 4 points?



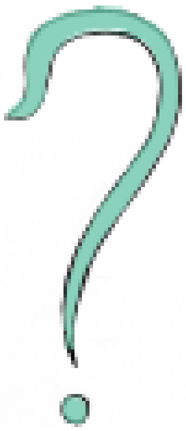


One additional point meant another line, could we have 2 parabolas here?

Curves

- Contstraints do not have to be on the function.
- They can also be on the derivative of it.

$$g(x) = a_2 \cdot x^2 + a_1 \cdot x + a_0 \qquad g'(x) = ?$$



- Constraints:

$$g(3) = 1$$

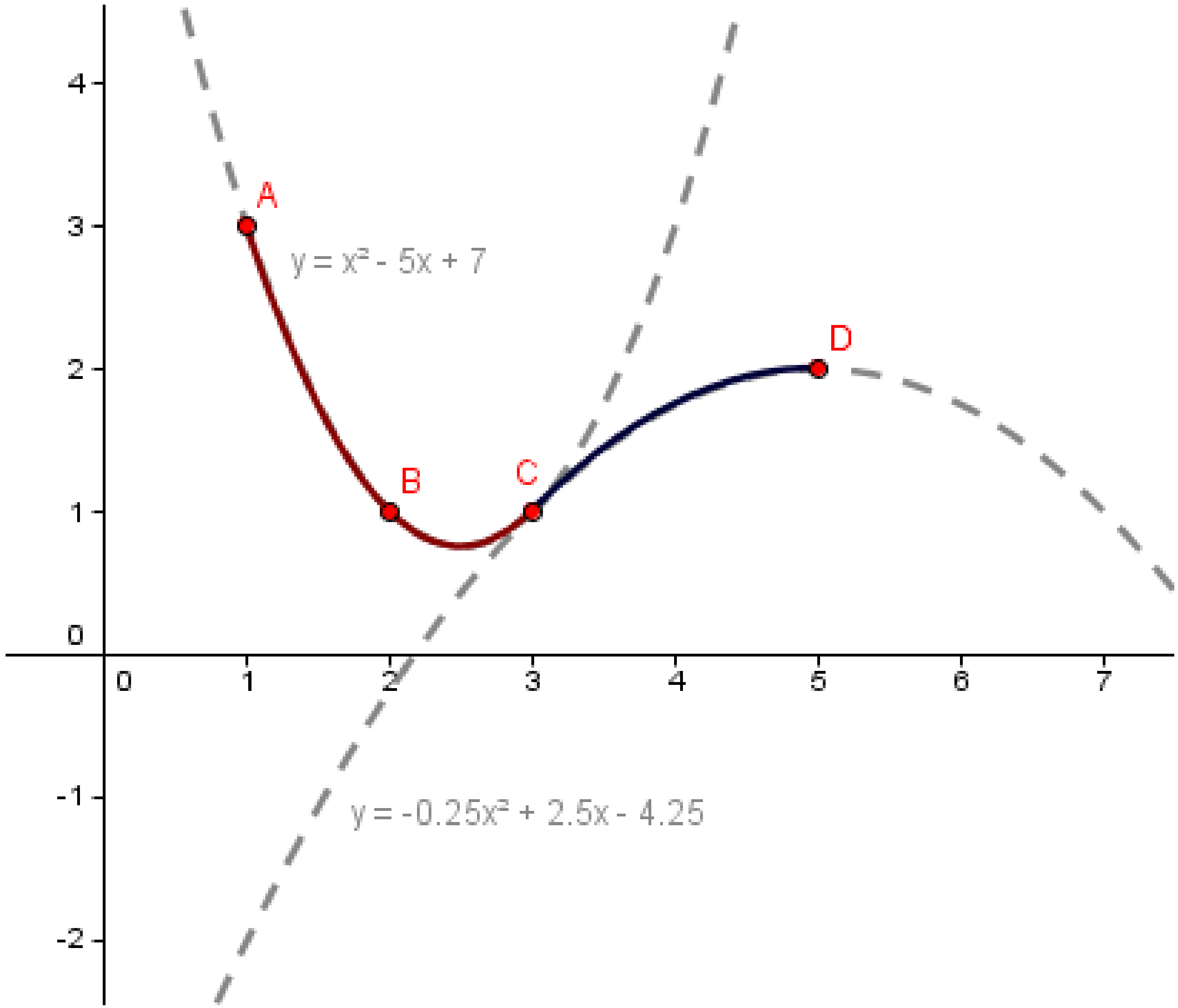
$$g(5) = 2$$

$$g'(3) = f'(3) = 1$$

http://www.wolframalpha.com/input/?i=9a%2B3b+%2B+c%3D1%2C+25a%2B5b%2Bc%3D2%2C+6a%2Bb%3D1

- 3 unknowns, 3 constraints, we can solve it.

Curves

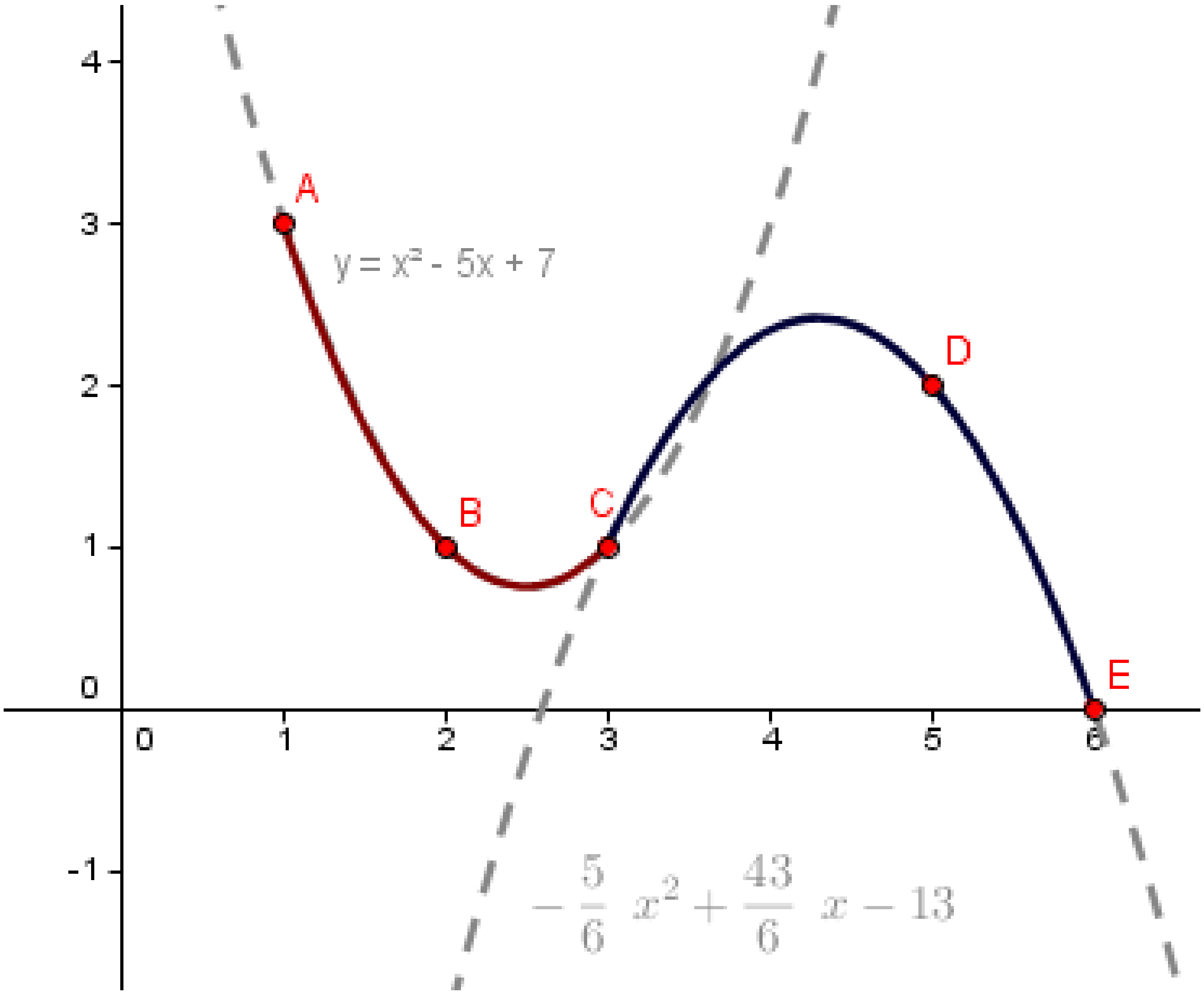


Smoothness

- What if we have 5 points and we put two parabolas through them without accounting for the derivative?

Smoothness

- That spline is not C^1 smooth.



Smoothness (continuity)

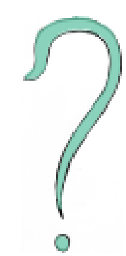
- Spline – many curves considered as one.
- C^n smoothness – the *n*-th derivative is continuous everywhere along the object.

- For **parametric curves**, we can also talk about:
  - G^n smoothness (geometric smoothness) – the *n*-th derivative can have sudden jumps in magnitude, but not the direction.

Smoothness (continuity)

- Different levels of smoothness:
  - C^0 – Curve itself is continuous
  - C^1 – First derivative (speed) is continuous
  - C^2 – Second derivative (acceleration) is continuous
- Often times in graphcis C^1 or C^2 smooth curves are enough.
- If we put quadratic curves together, so that the spline is C^1 smooth, how to get C^2 smoothness?

Find the second derivatives of our previous example...



Parametric Curves

- Implicit form: $f(x) = a_2 \cdot x^2 + a_1 \cdot x + a_0$
  - Good for testing points in a curve
  - Finding collisions
- Parametric form: $g(t) = (t + x_0,\ a_2 \cdot t^2 + y_0) = (x,\ y)$

$$x_0 = \frac{-a_1}{2 \cdot a_2}, \quad y_0 = f(x_0)$$

  - Good for generating points on the curve
  - What other parametric equations you know?

Parametric form of a quadratic polynomial: http://www.nabla.hr/PC-ParametricEqu2.htm

Parametric Curve Construction

- We want to find the vector coefficients a_i for a function of *t* (time), where $t \in [0..1]$.

Quadratic: $$curve(t) = a_0 + a_1 \cdot t + a_2 \cdot t^2$$

Cubic: $$curve(t) = a_0 + a_1 \cdot t + a_2 \cdot t^2 + a_3 \cdot t^3$$

- We need to have constraints. For example, the curve must interpolate a number of points.
- How many points we need?

Parametric Curve Construction

- For a quadradic curve, we need 3 points. This will give us 6 constraints, and we have 6 unknowns in 2D.
- What about in 3D? Cubic?
- Usually the system of constraints is written in a constraint matrix.

$$curve(0) = (1,\ 3) = p_0$$

$$curve(0.5) = (2,\ 1) = p_1$$

$$curve(1) = (3,\ 1) = p_2$$

Control points

Parametric Curve Construction

- Constraint matrix

In short: $C \cdot a = p$

How to find a ?

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0.25 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}$$

- Write out the equations to see, that this is exactly what we did before.
- Only now the a_i and p_i are vectors.

Parametric Curve Construction

- We can find $a = C^{-1} p$, the inverse constraint matrix C^{-1} is often denoted B and called the basis / blending matrix.
- Now we know the coefficients in:

$$curve(t) = a_0 + a_1 \cdot t + a_2 \cdot t^2$$

$$a_0 = b_{0,0}\, p_0 + b_{0,1}\, p_1 + b_{0,2}\, p_2$$
$$a_1 = b_{1,0}\, p_0 + b_{1,1}\, p_1 + b_{1,2}\, p_2$$
$$a_2 = b_{2,0}\, p_0 + b_{2,1}\, p_1 + b_{2,2}\, p_2$$

In our example:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \end{pmatrix}$$

Inverse (and other stuff) matrix calculator: http://www.bluebit.gr/matrix-calculator/calculate.aspx

Parametric Curve Construction

- Let's look at the entire curve:

$$curve(t) =$$
$$= b_{0,0}\, p_0 + b_{0,1}\, p_1 + b_{0,2}\, p_2 +$$
$$+ b_{1,0}\, p_0 t + b_{1,1}\, p_1 t + b_{1,2}\, p_2 t +$$
$$+ b_{2,0}\, p_0 t^2 + b_{2,1}\, p_1 t^2 + b_{2,2}\, p_2 t^2$$

- We can rewrite it as:

$$curve(t) = b_0(t) \cdot p_0 + b_1(t) \cdot p_1 + b_2(t) \cdot p_2$$

$$b_i(t) = b_{0,i} + b_{1,i} \cdot t + b_{2,i} \cdot t^2$$

Coefficients from one column of the matrix B.

The functions b_i are called **basis / blending functions**.

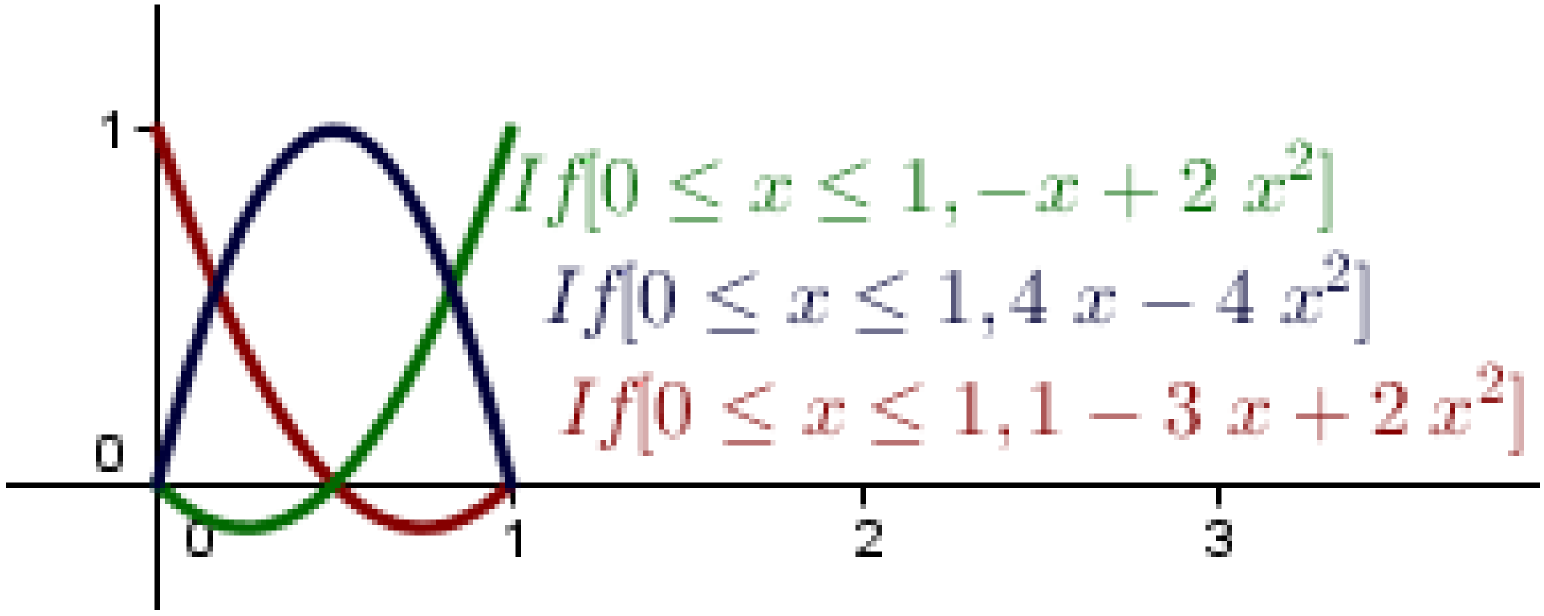
Parametric Curve Construction

- We have constructed a quadratic equation of time to interpolate our control points!
- Similar construction can be done for cubic equations and different other constraints (besides interpolation).

1) Pick a degree of the curve
2) Fix the parameters (*incl* control points)
3) Create the constraint matrix C
4) Find the basis matrix $B = C^{-1}$
5) Read the blending functions from the basis matrix

Blending Functions

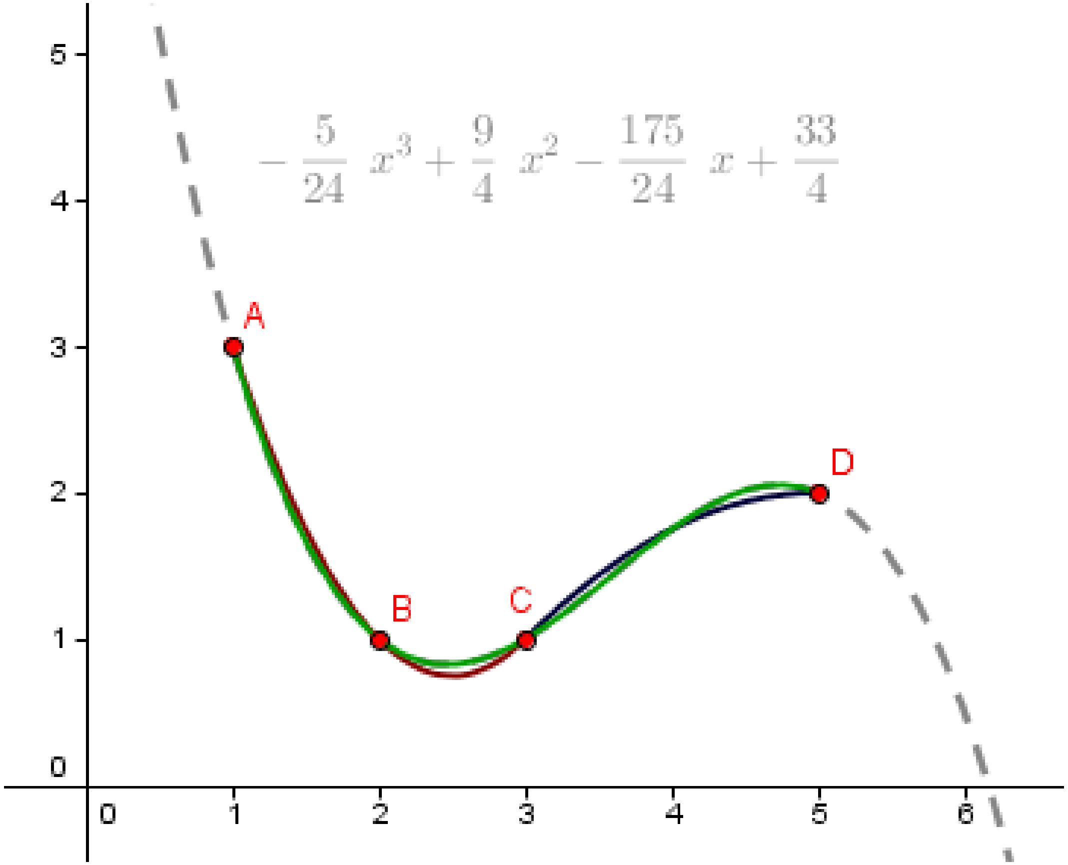
- Used to interpolate between the parameters.
- These would be the blending functions for interpolating a parabola between 3 points:



- Different constraints give different functions

Cubic not Quadratic

- In graphics, we usually want to use cubic polynomials, not quadratics.
- Cubic polynomials provide us with 4 possible constraints.
- Splines can achieve C^2 smoothness.



Hermite Spline

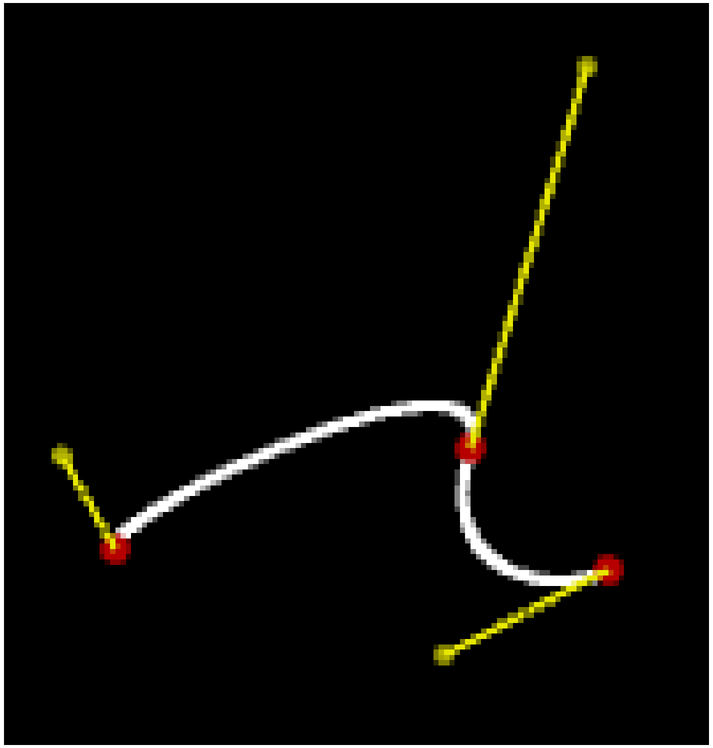
- The derivatives at the endpoints are parameters.
- Segments share the endpoints and derivatives.

$$curve(0) = p_0$$

$$curve'(0) = p_1$$

$$curve(1) = p_2$$

$$curve'(1) = p_3$$



Catmull-Rom Spline

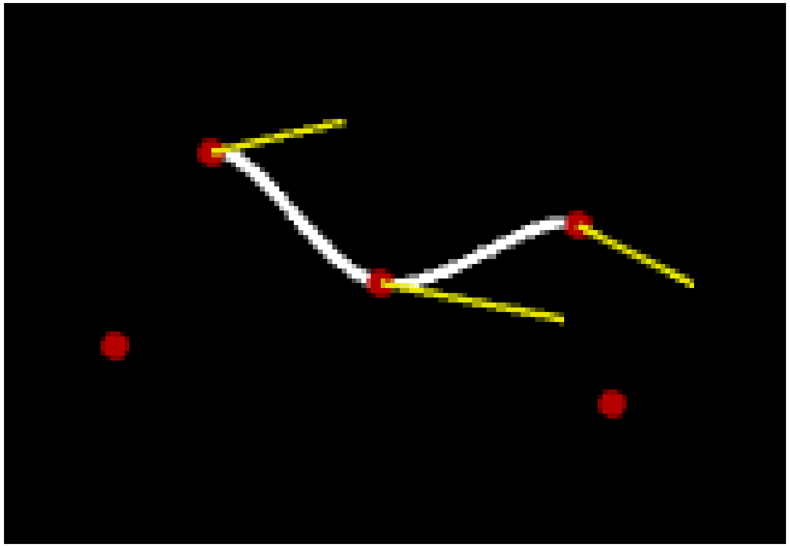
- We interpolate the p_1 and p_2.
- Dervatives are calculated using the other points.

$$curve'(0) = 0.5 \cdot (p_2 - p_0)$$

$$curve(0) = p_1$$

$$curve(1) = p_2$$

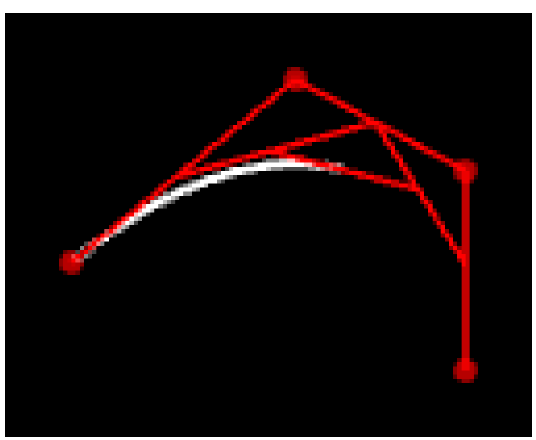
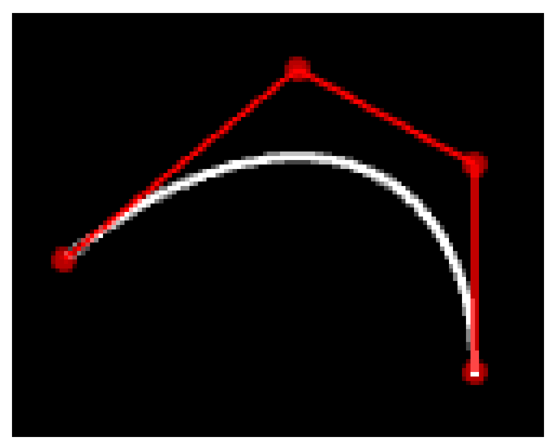
$$curve'(1) = 0.5 \cdot (p_3 - p_1)$$



- Only specify start and end derivatives, others are calculated.

Bezier Curve

- Could be constructed using the constraints and finding the blending functions.
- Could also be constructed in a procedural way:
  - Subdivide the lines connecting the control points, into proportions t and $(1-t)$.
  - Do it recusively until at last subdivision, which will give a point on the curve.

Bezier Curve

- That procedure is called De Casteljau's algorithm.
- The corresponding blending functions are called Bernstein polynomials.

Those you already used in the practice session.

$$b_{0,0}(t)=1$$

$$b_{0,1}(t)=1-t\,,\quad b_{1,1}(t)=t$$

$$b_{0,2}(t)=(1-t)^2\,,\quad b_{1,2}(t)=2\cdot t\cdot(1-t)\,,\quad b_{2,2}(t)=t^2$$

$$b_{i\,,\,degree}(t)=\binom{degree}{i}\cdot t^i\cdot(1-t)^{degree-i}$$

Bezier Curve

- Always inside the convex hull of the control points.
- Affine invariance – affine transformations on the control points, transform the curve itself correctly too.
- Sufficiently smooth splines can be constructed (Stärk's construction, we will see in the practice)
- Very widely used (*eg* font rendering)

Cubic Splines

- When constructing cubic splines, only 2 of the following properties can be satisfied at once:
  - Spline is C^2 smooth.
  - Spline interpolates the control points.
  - Spline has local control (changes in control points do not generally affect the entire curve).

- Hermite and Catmull-Rom – are not C^2 smooth.
- Bezier – does not interpolate the control points.

What did you learn today?

What more would you like to know?

Next time

Procedural Generation – *Jaanus Jaggo*