Computer Graphics

MTAT.03.015

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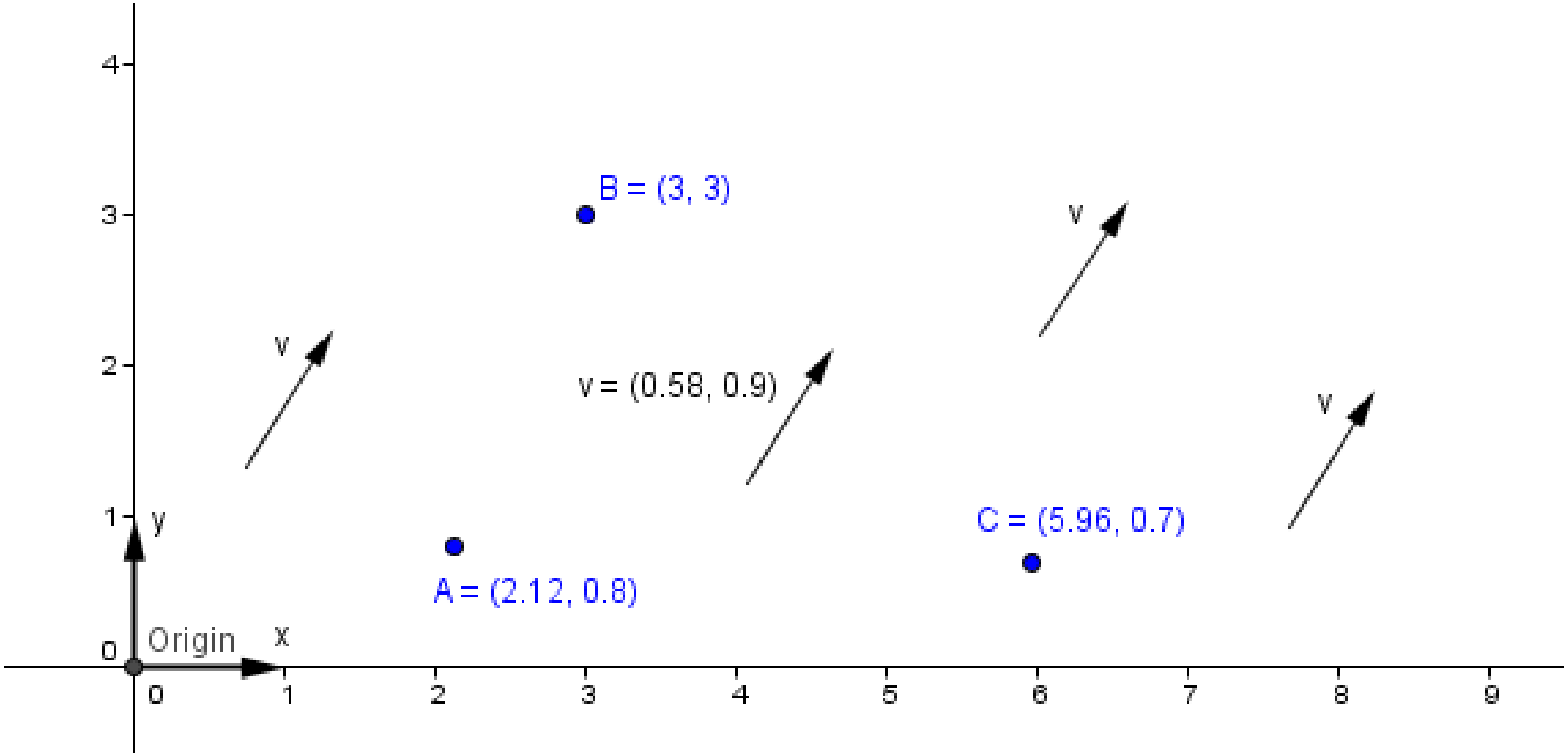
Points and Vectors

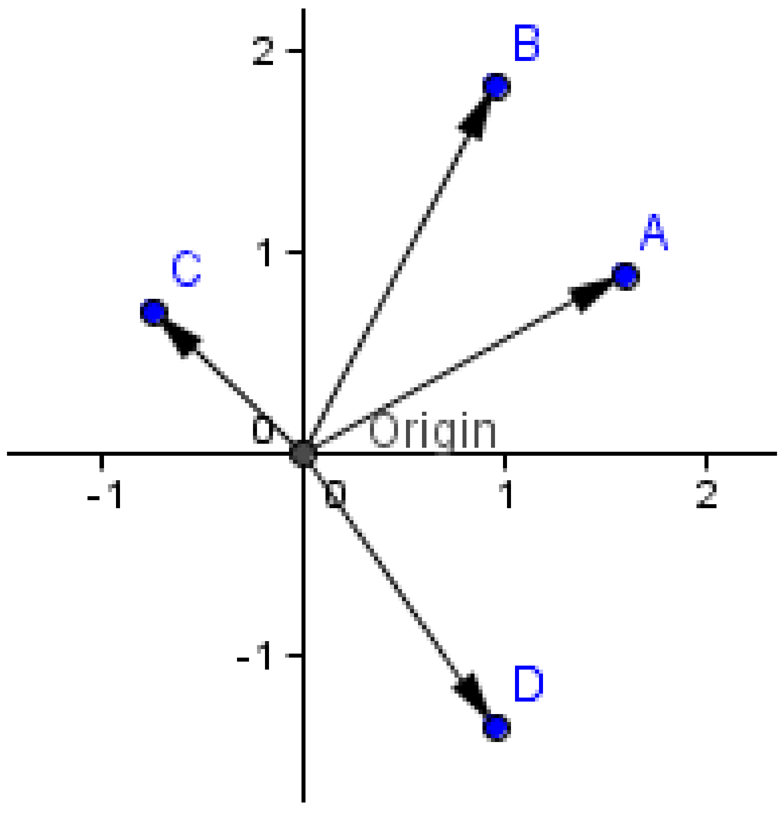
- In computer graphics we distinguish:
  - Point – a location in space (location vector, *kohavektor*)
  - Vector – a direction in space (direction vector, *suunavektor*)



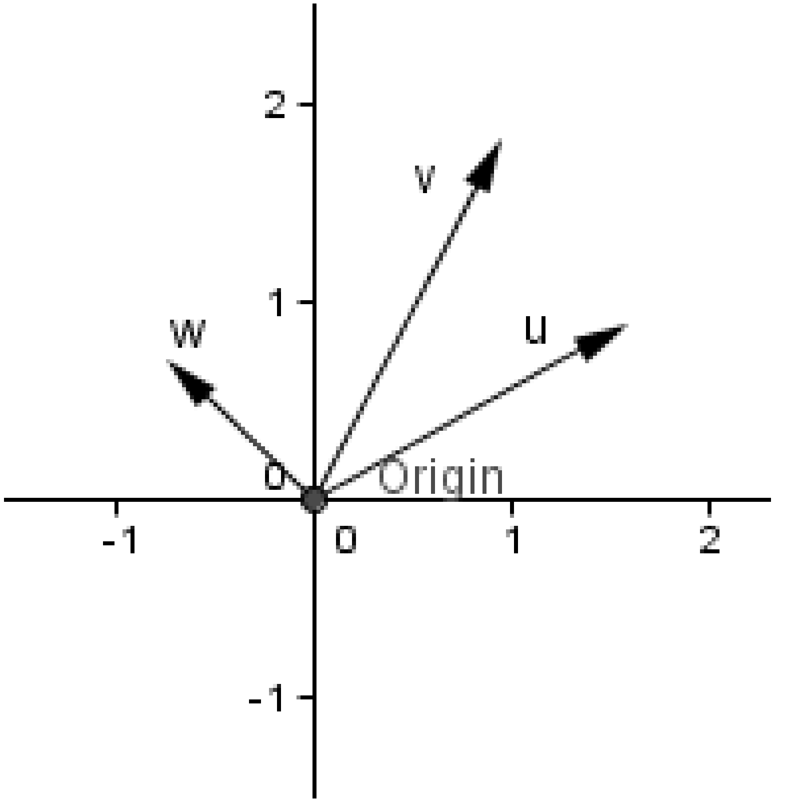
Points and Vectors

- Both are elements of a 2-, 3- or 4-dimensional **vector space** over the field $\mathbb{R}$
- More precisely, elements of a **coordinate space**.
- So both are *vectors* in terms of algebra.
- We distinguish them because some operations make sense for vectors, some for points.
- A space that contains both of them and defines an addition between a point and a vector is called an **affine space**.
- More precisely, an **Euclidean space**.

Points and Vectors



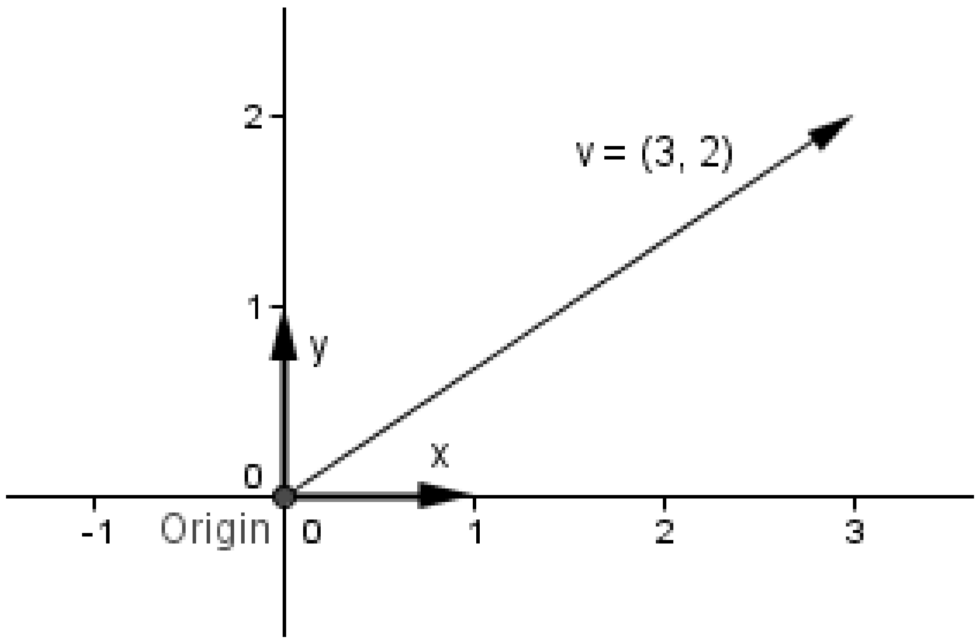
Vector space of points

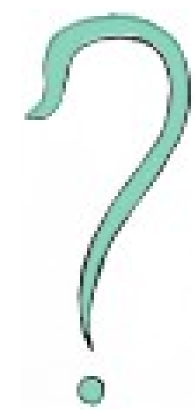


Vector space of vectors

Points and Vectors

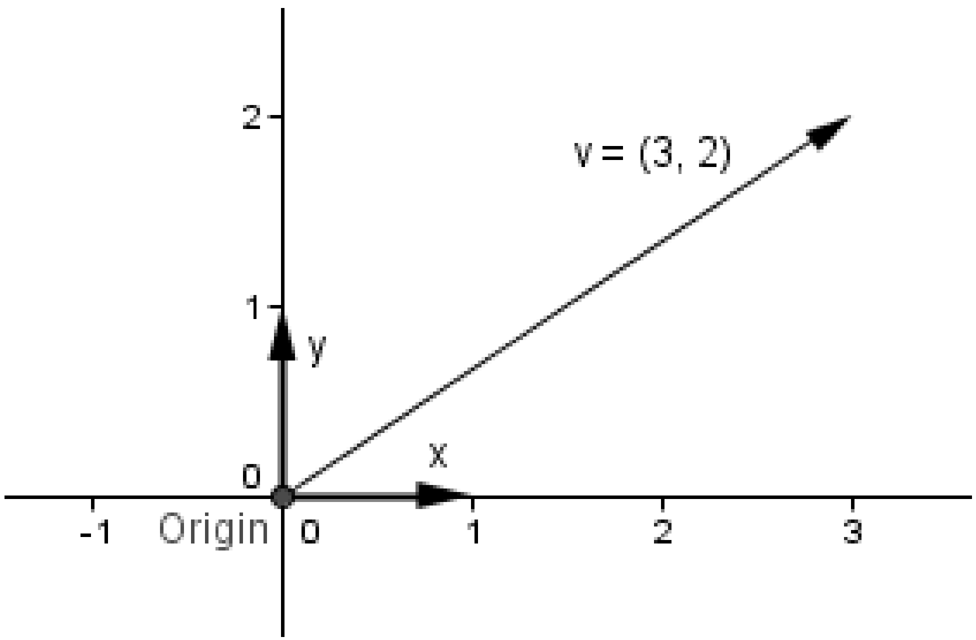
- Given a vector space over $\mathbb{R}^2$ with a **basis** and the **origin**, all the elements of the vector space can be represented as a...

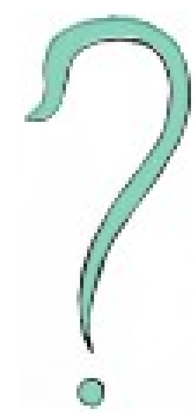




Points and Vectors

$$v = \alpha_0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

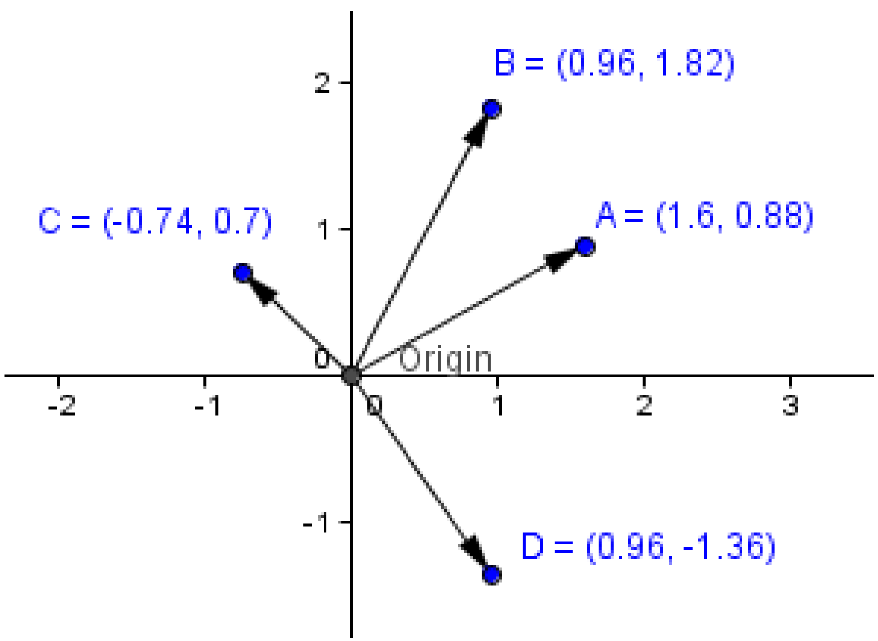




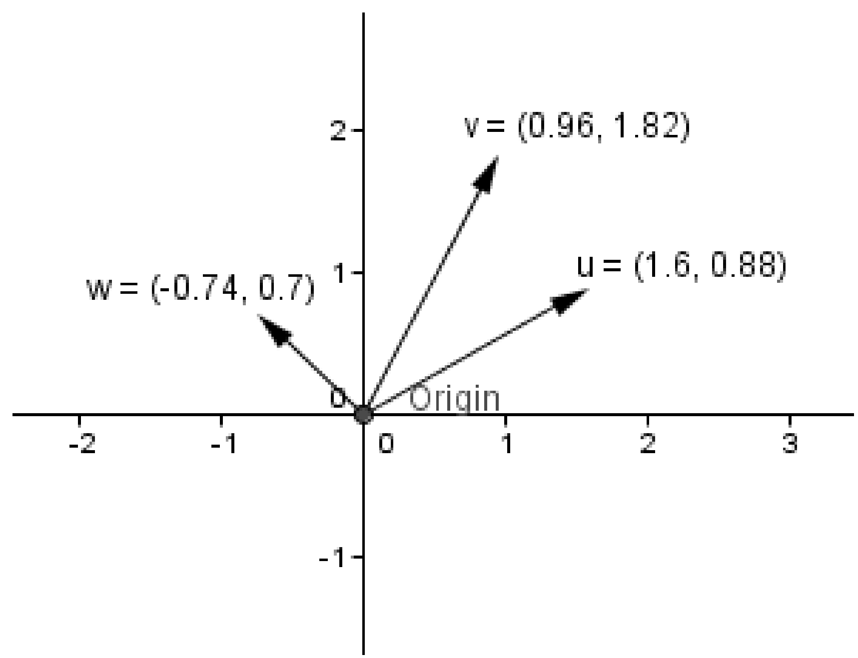
So, the scalar coefficients for our *v* would currently be?

Points and Vectors

- Because the elements of our vector space are *n*-tuples, we can call it a **coordinate space**.



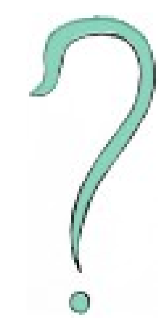
Coordinate space of points



Coordinate space of vectors

Points and Vectors

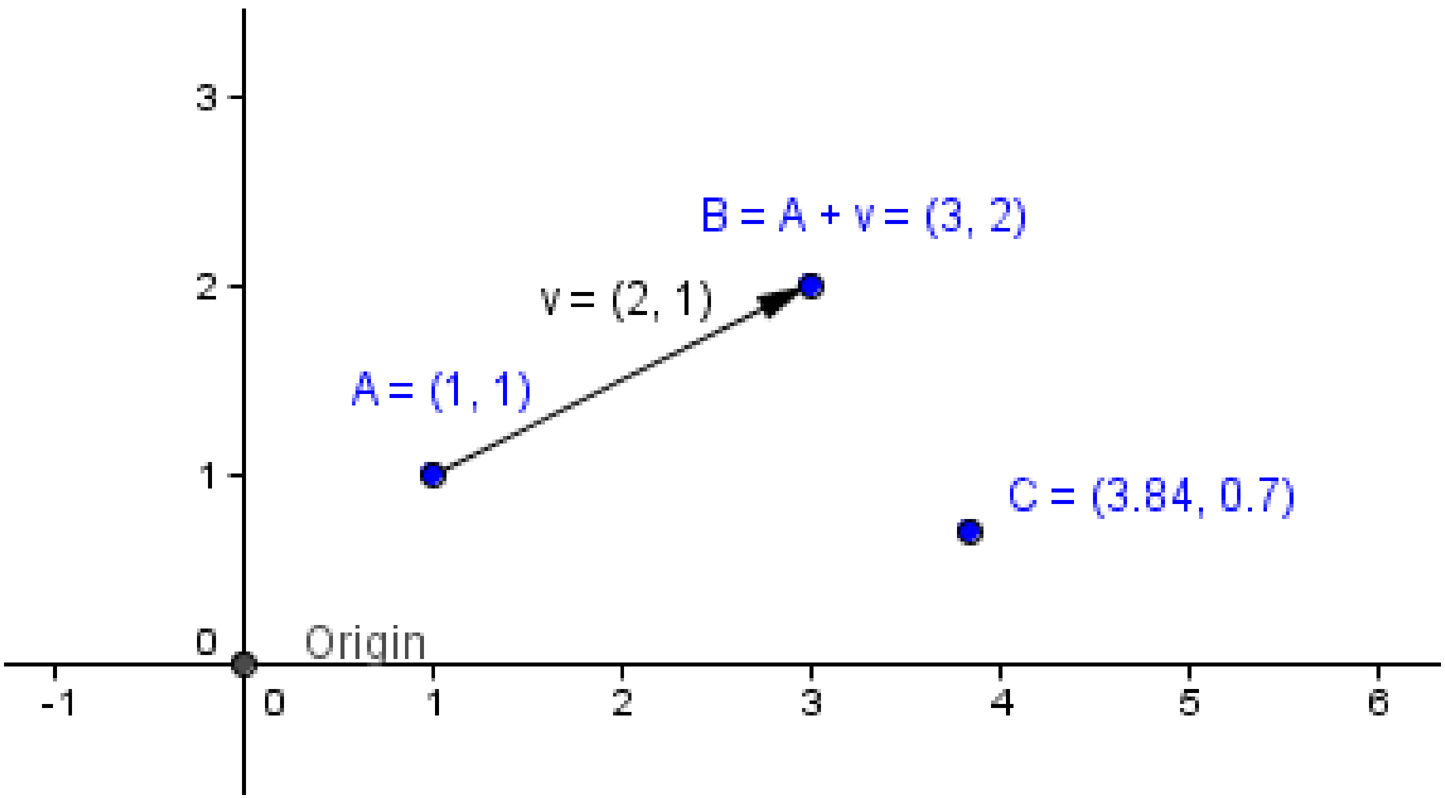
- Besides just doing operations between points, or between vectors, we want to do operations between them.



- Or do we? Can you think of an operation we would want to do between a point and a vector?

Points and Vectors

- When we put those two spaces together, we get an **affine space**.



Points and Vectors

- Is this a point or a vector?

$$x = \begin{pmatrix} 3 & 2 \end{pmatrix}$$

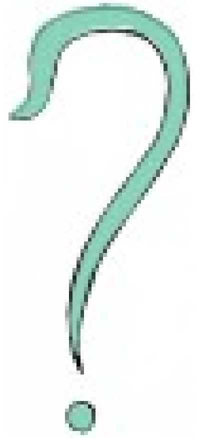
- What about this?

$$x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Points and Vectors

- **Row-major** and **column-major** formats.
- Which is which?
- How to get from one to another?

$$x = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad x = \begin{pmatrix} 3 & 2 \end{pmatrix}$$



Sometimes the reader is expected to guess which one is used based on the context.

Good explanation:
http://www.scratchapixel.com/lessons/mathematics-physics-for-computer-graphics/geometry/row-major-vs-column-major-vector

Points and Vectors

- **Homogeneous coordinates** – a notation where we add an additional coordinate to distinguish between points and vectors.

2D $$p = \begin{pmatrix} x & y & z \end{pmatrix} = \left(\frac{x}{z}, \frac{y}{z} \right)$$

3D $$p = \begin{pmatrix} x & y & z & w \end{pmatrix} = \begin{pmatrix} \frac{x}{w} & \frac{y}{w} & \frac{z}{w} \end{pmatrix}$$

Points and Vectors

- In 2D homogeneous coordinates:
  - Point

$$p = \begin{pmatrix} x & y & z \end{pmatrix} \qquad z \neq 0$$

  - Vector

$$p = \begin{pmatrix} x & y & z \end{pmatrix} \qquad z = 0$$

- Vector is a point located in infinity

Points and Vectors

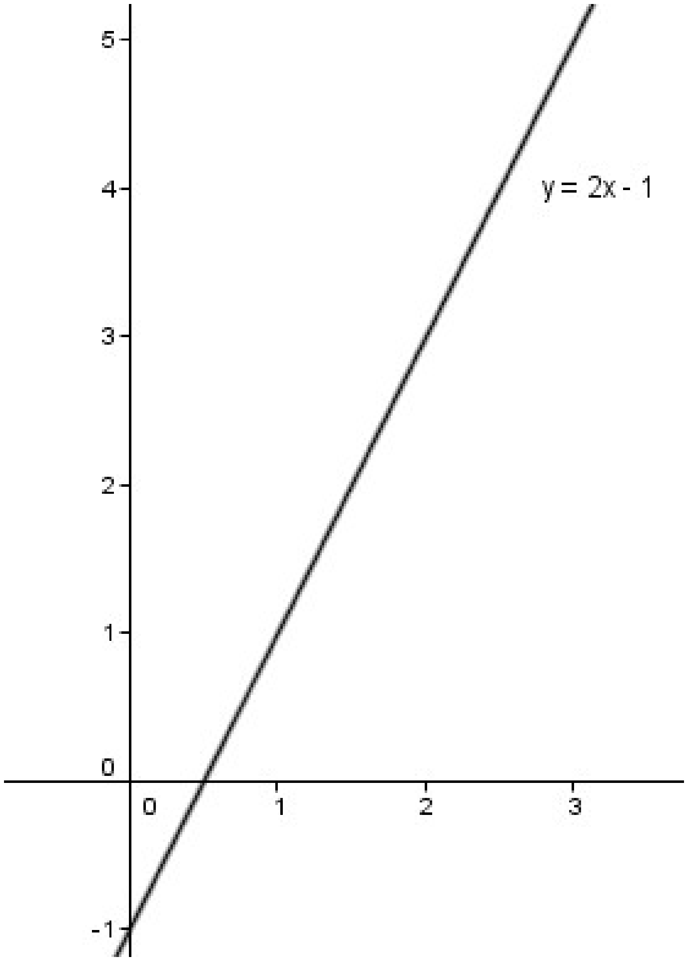
- What should *z* be if you want to define a point located at *(x, y)*?

Normalized point

- How does addition work now?
- Addition between two vectors?
- Addition between two points?
- Subtraction of points?

Line

- You probably know about the **implicit** line equation: $y = a \cdot x + b$
  It defines the relationship between the coordinates.
- Can also be used to test if a given point is on the line or not. How?

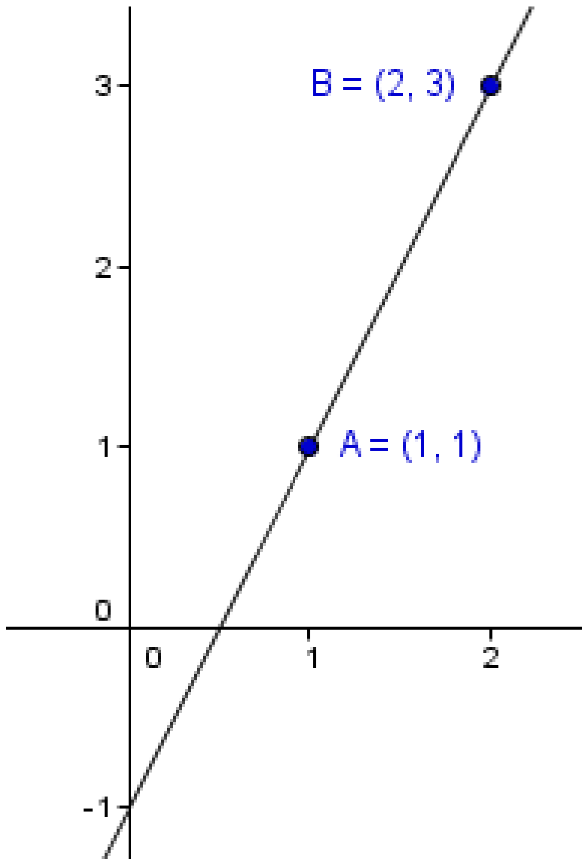


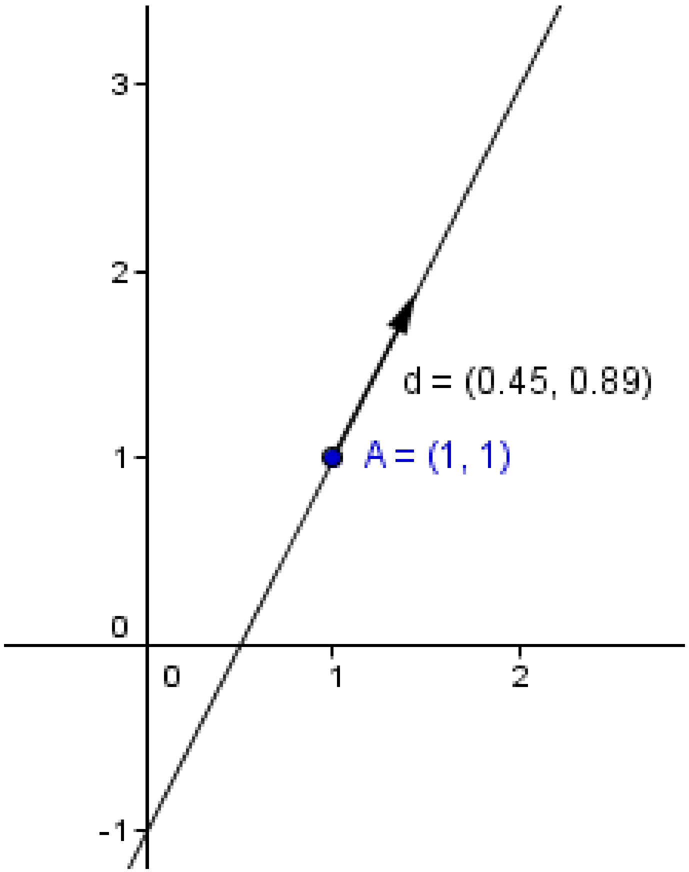
Line

- How can we represent a line in our affine space?
- We do not know *a* (the slope) or *b* (y-intercept).
- What would we need to know to represent a line?

Line

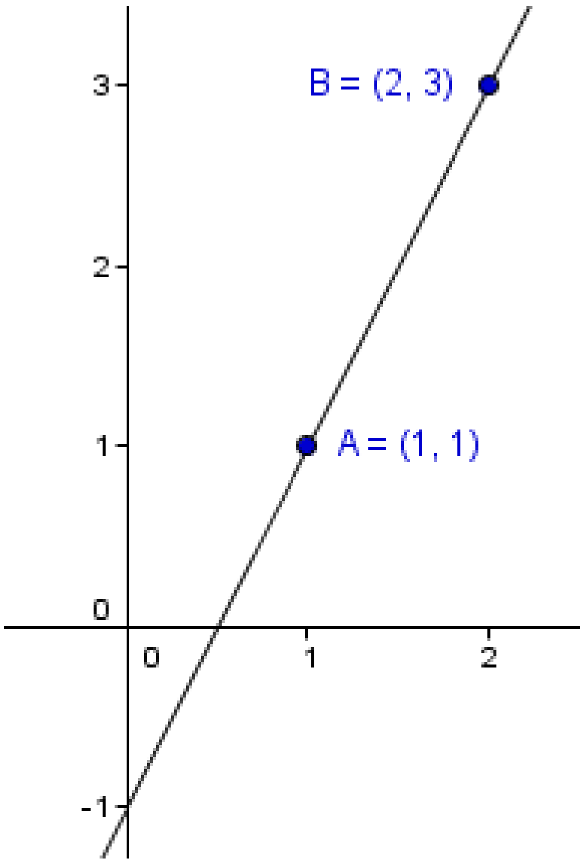
- Two points that the line passes
- One point and a direction vector

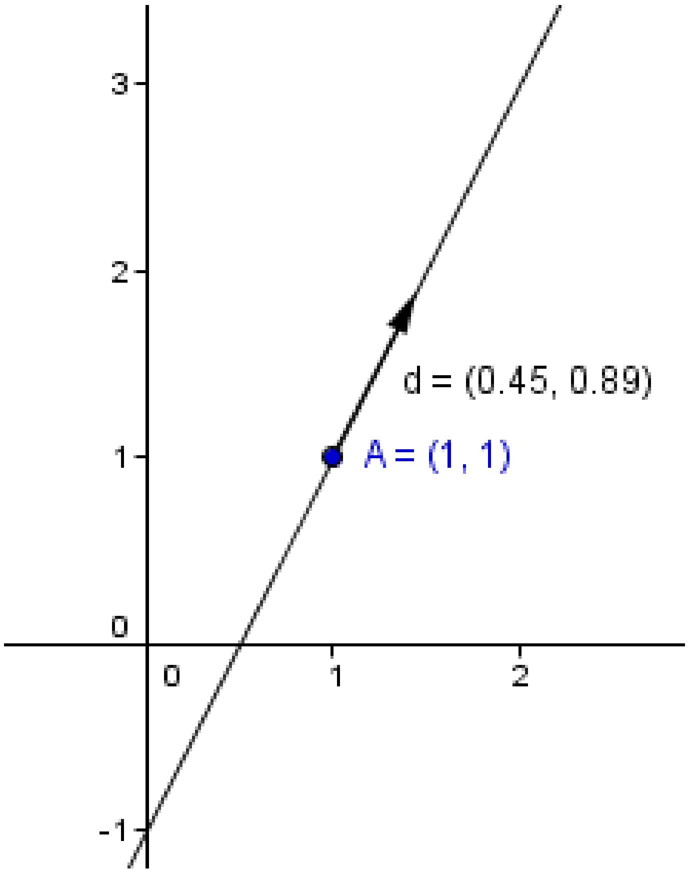




Line

$$line = (1-\alpha)\cdot A + \alpha \cdot B \qquad line = A + \alpha \cdot d$$



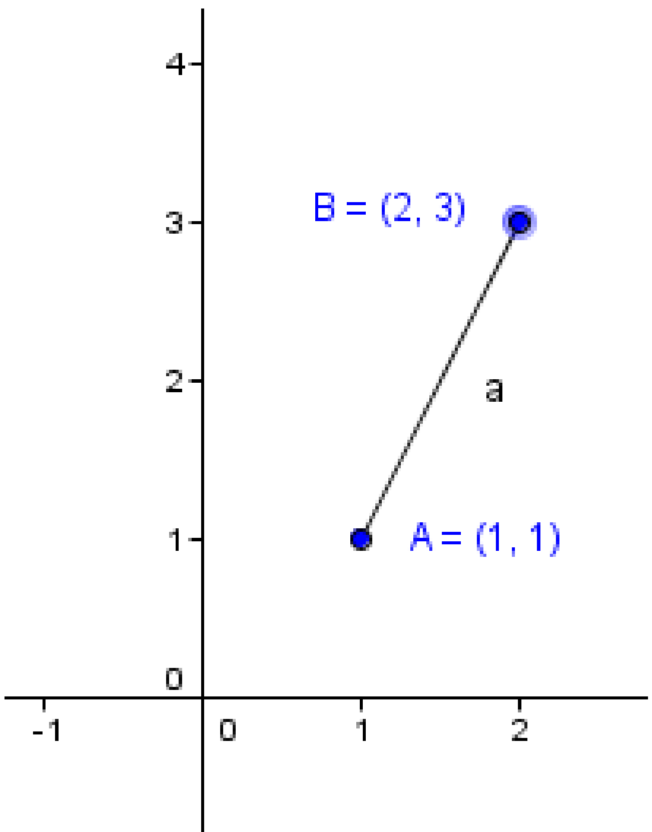


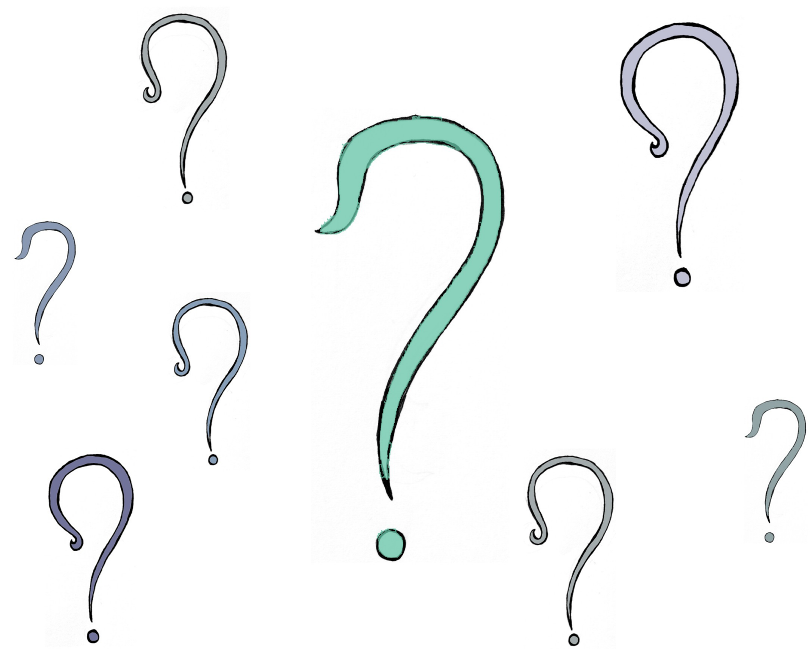
$$d = B - A$$

Line Segment

- Knowing that: $line = (1-\alpha)\cdot A + \alpha\cdot B$

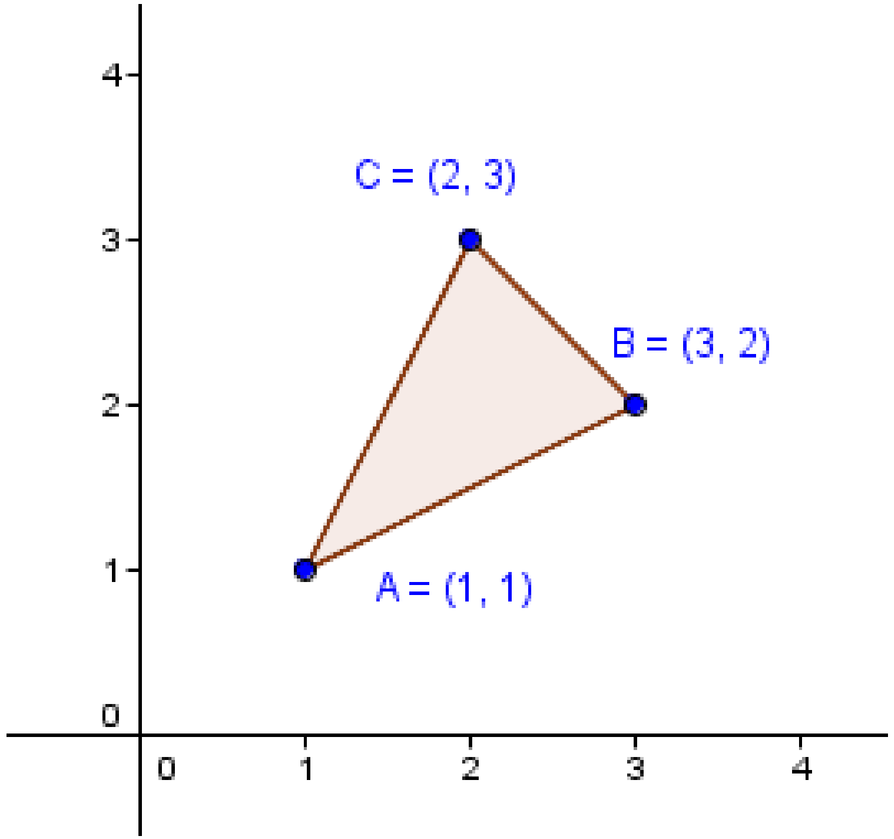
How to represent a line segment?





Triangle

- How about a triangle?



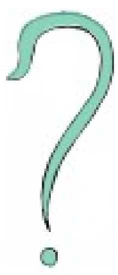
Barycentric Coordinates

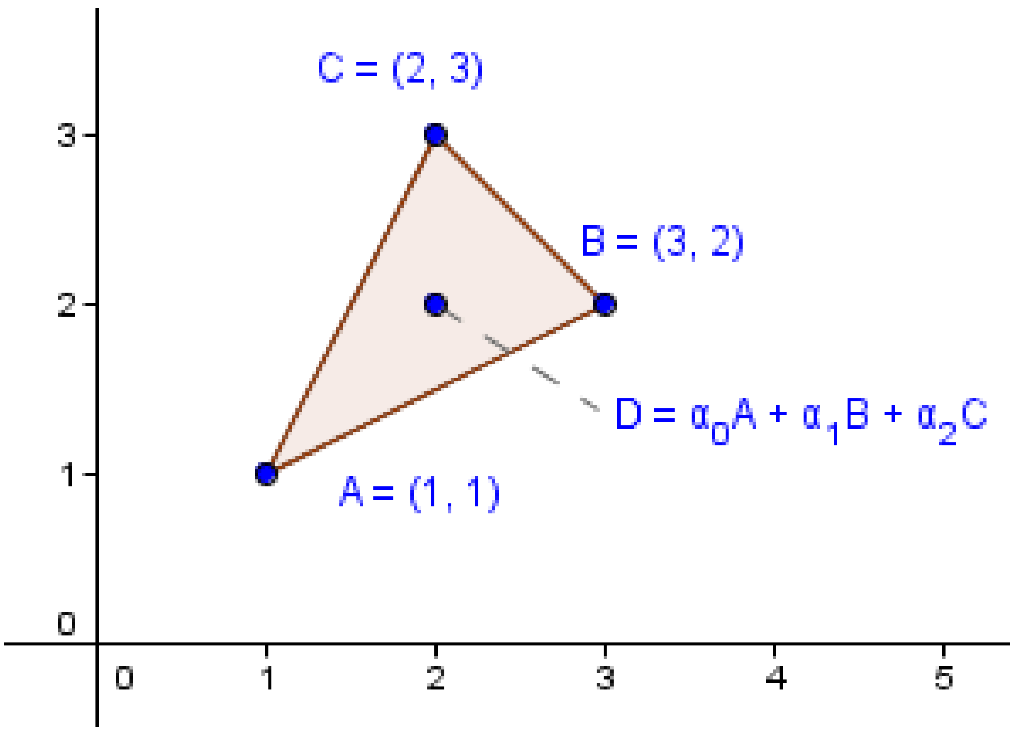
- The coefficients of a **convex combination** of the vertices are the **Barycentric coordinates** of all the points inside the triangle.

$$triangle = \alpha_0 \cdot A + \alpha_1 \cdot B + \alpha_2 \cdot C$$

$$\alpha_i \geq 0, \quad \alpha_0 + \alpha_1 + \alpha_2 = 1$$

What are the coordinates of the vertices in the Barycentric system?





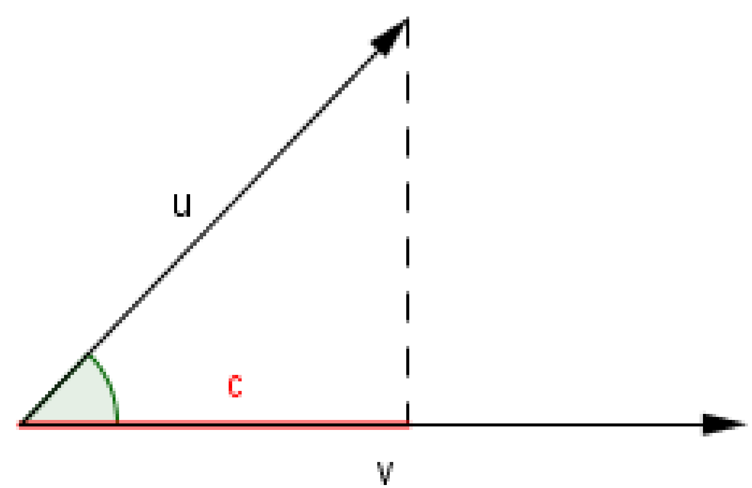
Find them for other easy points.

Dot Product

- Useful operation between vectors. Why?
- Definition
  - Geometric: $$u \cdot v = \|u\| \cdot \|v\| \cdot \cos(\angle\, uv)$$
  - Algebraic: $$u \cdot v = u_0 \cdot v_0 + u_1 \cdot v_1 + u_2 \cdot v_2$$
- Also called: scalar product, inner product
- *Skalaarkorrutis*

Scalar Projection

- Dot product can be used to project one vector onto another.
- Scalar projection of u onto v is: $$c = u \cdot \frac{v}{\|v\|} = u \cdot \hat{v}$$
- It gives you the length, how much $\hat{v}$ you have to take in order to reach the orthogonal projection point of u.



Eg The Gram–Schmidt process

Cross Product

- Returns a vector orthogonal to the operands.
- Definition

$$u \times v = n \cdot \|u\| \cdot \|u\| \cdot \sin(\angle uv)$$

  - Geometric
  - Algebraic

$$u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

- Also called: vector product
- *Vektorkorrutis*

Direction of the result depends on the handedness of the coordinate system.

Scalar Triple Product

- Definition: $u \cdot (v \times w)$
- Useful in solving a system of equations of vectors, because:

$$u \cdot (v \times w) = \begin{vmatrix} u_0 & u_1 & u_2 \\ v_0 & v_1 & v_2 \\ w_0 & w_1 & w_2 \end{vmatrix}$$

- We can see this in Basic II, with triangle-ray intersection testing.
- *Segakorrutis*.

What was important for you today?

What more would you like to know?

Next time: Transformations

(scale, shear, rotate, translate)