Computer Graphics

MTAT.03.015

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Points and Vectors

- In computer graphics we distinguish:
  - Point – a location in space (location vector, *kohavektor*)
  - Vector – a direction in space (direction vector, *suunavektor*)
Points and Vectors

• Both are elements of a 2-, 3- or 4-dimensional vector space over the field $\mathbb{R}$.
• More precisely, elements of a coordinate space.
• So both are vectors in terms of algebra.
• We distinguish them because some operations make sense for vectors, some for points.
• A space that contains both of them and defines an addition between a point and a vector is called an affine space.
• More precisely, an Euclidean space.
Points and Vectors

Vector space of points

Vector space of vectors
Points and Vectors

- Given a vector space over $\mathbb{R}^2$ with a basis and the origin, all the elements of the vector space can be represented as a...
Points and Vectors

\[ \mathbf{v} = \alpha_0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

So, the scalar coefficients for our \( \mathbf{v} \) would currently be?
Points and Vectors

- Because the elements of our vector space are $n$-tuples, we can call it a coordinate space.

Coordinate space of points

Coordinate space of vectors
Points and Vectors

• Besides just doing operations between points, or between vectors, we want to do operations between them.

• Or do we? Can you think of an operation we would want to do between a point and a vector?
Points and Vectors

- When we put those two spaces together, we get an affine space.
Points and Vectors

• Is this a point or a vector?

\[ x = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \]

• What about this?

\[ x = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \]
Points and Vectors

- Row-major and column-major formats.
- Which is which?
- How to get from one to another?

\[ x = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad x = \begin{pmatrix} 3 & 2 \end{pmatrix} \]

Sometimes the reader is expected to guess which one is used based on the context.

Good explanation:
Points and Vectors

- **Homogeneous coordinates** – a notation where we add an additional coordinate to distinguish between points and vectors.

\[ p = (x \ y \ z) = \left( \frac{x}{z}, \frac{y}{z} \right) \]

\[ p = (x \ y \ z \ w) = \left( \frac{x}{w}, \frac{y}{w}, \frac{z}{w} \right) \]
Points and Vectors

• In 2D homogeneous coordinates:
  • Point
    \[ p = (x, y, z) \quad z \neq 0 \]
  • Vector
    \[ p = (x, y, z) \quad z = 0 \]

• Vector is a point located in infinity
Points and Vectors

• What should $z$ be if you want to define a point located at $(x, y)$?

• How does addition work now?

• Addition between two vectors?

• Addition between two points?

• Subtraction of points?
You probably know about the **implicit** line equation: \( y = a \cdot x + b \)

It defines the relationship between the coordinates.

Can also be used to test if a given point is on the line or not. How?
Line

• How can we represent a line in our affine space?
• We do not know \( a \) (the slope) or \( b \) (y-intercept).
• What would we need to know to represent a line?
Line

- Two points that the line passes
- One point and a direction vector
Line

\[ \text{line} = (1 - \alpha) \cdot A + \alpha \cdot B \quad \text{line} = A + \alpha \cdot d \]

\[ d = B - A \]
Line Segment

- Knowing that: \( \text{line} = (1 - \alpha) \cdot A + \alpha \cdot B \)

How to represent a line segment?
Triangle

- How about a triangle?
Barycentric Coordinates

• The coefficients of a convex combination of the vertices are the **Barycentric coordinates** of all the points inside the triangle.

\[
\text{triangle} = \alpha_0 \cdot A + \alpha_1 \cdot B + \alpha_2 \cdot C
\]

\[
\alpha_i \geq 0, \quad \alpha_0 + \alpha_1 + \alpha_2 = 1
\]

What are the coordinates of the vertices in the Barycentric system?

Find them for other easy points.
Dot Product

• Useful operation between vectors. Why?

• Definition
  
  • Geometric:  \[ u \cdot v = \|u\| \cdot \|v\| \cdot \cos(\angle uv) \]
  
  • Algebraic:  \[ u \cdot v = u_0 \cdot v_0 + u_1 \cdot v_1 + u_2 \cdot v_2 \]

• Also called: scalar product, inner product

• *Skalaarkorrutis*
Scalar Projection

• Dot product can be used to project one vector onto another.
• Scalar projection of $u$ onto $v$ is: $c = u \cdot \frac{v}{\|v\|} = u \cdot \hat{v}$
• It gives you the length, how much $\hat{v}$ you have to take in order to reach the orthogonal projection point of $u$.  

*Eg The Gram–Schmidt process*
Cross Product

- Returns a vector orthogonal to the operands.

- Definition
  \[ u \times v = n \cdot \|u\| \cdot \|u\| \cdot \sin(\angle uv) \]

- Geometric

- Algebraic
  \[ u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \]

- Also called: vector product

- *Vektorkorrutilis*

Direction of the result depends on the handedness of the coordinate system.
Scalar Triple Product

- Definition: \( u \cdot (v \times w) \)

- Useful in solving a system of equations of vectors, because:

\[
\begin{vmatrix}
  u_0 & u_1 & u_2 \\
  v_0 & v_1 & v_2 \\
  w_0 & w_1 & w_2
\end{vmatrix}
\]

- We can see this in Basic II, with triangle-ray intersection testing.

- \textit{Segakorrutis}. 

\[u \cdot (v \times w) = \begin{vmatrix}
  u_0 & u_1 & u_2 \\
  v_0 & v_1 & v_2 \\
  w_0 & w_1 & w_2
\end{vmatrix} \]
What was important for you today?

What more would you like to know?

Next time: Transformations
(scale, shear, rotate, translate)