Computer Graphics
MTAT.03.015

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Study IT in .ee
The Road So Far...

Last week:
- Construct geometry
- Define transformations
- Assign material properties

This week:
- Vertex Transformations
- Culling & Clipping
- Determine front-facing triangles
- Determine which vertices are visible
- Rasterization
- Fill the triangle with fragments
- Fragment Shading
- Calculate correct color values
- Visibility Tests
- Blending
- Is the fragment visible?
- Blend together multiple fragments
Transformations

• Watch the Computerphile video, try to find out:
  1) Why are we using matrices?

The True Power of the Matrix (Transformations in Graphics) – Computerphile
https://www.youtube.com/watch?v=vQ60rFwh2ig
Transformations

• Watch the Computerphile video, try to find out:
  1) Why are we using matrices?
  2) Where do the homogeneous coordinates come in?

The True Power of the Matrix (Transformations in Graphics) – Computerphile
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Linear Transformations

- Also called *linear mapping*, *linear function*
Linear Transformations

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- Transforms a vector space $V$ into a vector space $W$, while preserving addition and scalar multiplication
Linear Transformations

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- Transforms a vector space $V$ into a vector space $W$, while preserving addition and scalar multiplication
- Satisfies: $f(\alpha \cdot v + \beta \cdot u) = \alpha \cdot f(v) + \beta \cdot f(u)$
Linear Transformations

- Also called *linear mapping*, *linear function*
- Transforms a vector space $V$ into a vector space $W$, while preserving addition and scalar multiplication
- Satisfies: $f(\alpha \cdot v + \beta \cdot u) = \alpha \cdot f(v) + \beta \cdot f(u)$
- In 3D: $\alpha, \beta \in \mathbb{R}$, $u, v \in \mathbb{R}^3$
Linear Transformations

- Take our vector space of points
Linear Transformations

- Take our vector space of points
- Take for example a point $p = (2, 1)$
Linear Transformations

- Take our vector space of points
- Take for example a point \( \mathbf{p} = (2, 1) \)
- Try mappings:
  1) \( f(p) = (p_x, p_y) \)
  2) \( f(p) = (2 \cdot p_x, p_y) \)
  3) \( f(p) = (p_x, 2 \cdot p_y) \)
  4) \( f(p) = (2 \cdot p_x, 2 \cdot p_y) \)
Linear Transformations

• From Algebra you know that all linear transformations can be represented as matrices.

Linear transformation $\rightarrow$ Matrix
Linear Transformations

• From Algebra you know that all linear transformations can be represented as matrices.

• Every matrix also gives you a linear transformation.

Linear transformation $\rightarrow$ Matrix

Linear transformation $\leftarrow$ Matrix
Linear Transformations

- What would be the matrices for the linear transformations we just saw?

\[
f(p) = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}
\]

\[
f(p) = (p_x, p_y)
\]

\[
f(p) = (2 \cdot p_x, p_y)
\]

\[
f(p) = (p_x, 2 \cdot p_y)
\]

\[
f(p) = (2 \cdot p_x, 2 \cdot p_y)
\]
• Stretches or shrinks the space

2D
\[
\begin{pmatrix}
a_x & 0 \\
0 & a_y
\end{pmatrix}
\]
\(a_x - x\)-axis scale factor
\(a_y - y\)-axis scale factor

3D
\[
\begin{pmatrix}
a_x & 0 & 0 \\
0 & a_y & 0 \\
0 & 0 & a_z
\end{pmatrix}
\]
\(a_x - x\)-axis scale factor
\(a_y - y\)-axis scale factor
\(a_z - z\)-axis scale factor
Scale

- Transformations can be easily understood, if we see what they do with the standard basis.
Scale

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Scale

• Transformations can be easily understood, if we see what they do with the standard basis

• Furthermore, one can read the transformed standard basis from the columns of the transformation
Shear

- Shear-x, shear-y
- Tilts one of the axes

Shear-x or shear-y?
Matrix?
Shear

- Shear-y, we tilt the \( x \) basis vector parallel to \( y \) by angle \( \varphi \) counterclockwise

\[
\begin{pmatrix}
1 & 0 \\
\tan(\varphi) & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
x \\
y + \tan(\varphi) \cdot x
\end{pmatrix}
\]

- Shear-x, we tilt the \( y \) basis vector parallel to \( x \) by angle \( \varphi \) clockwise

\[
\begin{pmatrix}
1 & \tan(\varphi) \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
x + \tan(\varphi) \cdot y \\
y
\end{pmatrix}
\]

What about in 3D?
Rotation

- We want to keep the basis vectors on the unit-circle.

Can you derive the matrix?
Rotation

\[ e'_0 = (|a|, |b|) = (\cos(\alpha), \sin(\alpha)) \]
\[ e'_1 = (|a'|, |b'|) = (-\sin(\alpha), \cos(\alpha)) \]

\[ \cos(\alpha) = \frac{|a|}{|e'_0|} = \frac{|a|}{1} = |a| \]
Rotation

- Rotates around an axis (or a direction)

\[
2D \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad \alpha \text{ – Positive angle to rotate by}
\]
Rotation

- Rotates around an axis (or a direction)

\[
\begin{pmatrix}
\cos(\alpha) & -\sin(\alpha) \\
\sin(\alpha) & \cos(\alpha)
\end{pmatrix}
\]

\(\alpha\) – Positive angle to rotate by

- Similar matrices that rotate around each axis.
Rotation

- Rotates around an axis (or a direction)

\[
\begin{pmatrix}
\cos(\alpha) & -\sin(\alpha) \\
\sin(\alpha) & \cos(\alpha)
\end{pmatrix}
\]

\(\alpha\) – Positive angle to rotate by

- Similar matrices that rotate around each axis.

- What about rotation around an arbitrary direction?
Linear Transformations

Defined geometry
Linear Transformations

Scale
Linear Transformations
Linear Transformations

Rotation
Linear Transformations
Linear Transformations

Shear
Linear Transformations

Will these be enough?
Translation

- Imagine a 1D world located at $y=1$ line in 2D.
Translation

- Imagine a 1D world located at $y=1$ line in 2D.
Translation

- Imagine a 1D world located at $y=1$ line in 2D.

- Notice that all the points are in the form: $(x, 1)$
Translation

• How to transform the 2D space so that stuff in the 1D hyperplane $y=1$ moves an equal amount?
Translation

- Shear-x by $\tan(45^\circ) = 1$

- Shear-x with $\tan(63.4^\circ) = 2$
Translation

- Affine transformation in the current space, linear shear transformation in 1 dimension higher space.

\[
\begin{align*}
\text{2D} & \quad \begin{pmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + x_t \\ y + y_t \\ 1 \end{pmatrix} \\
\text{Shear-xy} & \\
\text{3D} & \quad \begin{pmatrix} 1 & 0 & 0 & x_t \\ 0 & 1 & 0 & y_t \\ 0 & 0 & 1 & z_t \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + x_t \\ y + y_t \\ z + z_t \\ 1 \end{pmatrix} \\
\text{Shear-xyz} & \\
\text{1D} & \quad \begin{pmatrix} 1 & x_t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} x + x_t \\ 1 \end{pmatrix} \\
\text{Shear-x} &
\end{align*}
\]
Transformations

• This together gives us a very good toolset to transform our geometry as we wish.

\[
\begin{bmatrix}
a & b & c & x_t \\
d & e & f & y_t \\
g & h & i & z_t \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1 \\
\end{bmatrix}
= 
\begin{bmatrix}
ax + by + cz + x_t \\
dx + ey + fz + y_t \\
gx + hy + iz + z_t \\
1 \\
\end{bmatrix}
\]
Transformations

- This together gives us a very good toolset to transform our geometry as we wish.

Linear transformations

\[
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
  0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_t \\
  y_t \\
  z_t \\
  1
\end{bmatrix}
\]

\[
\begin{bmatrix}
  ax + by + cz + x_t \\
  dx + ey + fz + y_t \\
  gx + hy + iz + z_t \\
  1
\end{bmatrix}
\]
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\begin{pmatrix}
x \\
y \\
z \\
1
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= 
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ax + by + cz + x_t \\
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\end{pmatrix}
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\[
\begin{pmatrix}
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\end{pmatrix}
\begin{pmatrix}
    x \\
    y \\
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\end{pmatrix} =
\begin{pmatrix}
    ax + by + cz + x_t \\
    dx + ey + fz + y_t \\
    gx + hy + iz + z_t \\
    1
\end{pmatrix}
\]
Multiple Transformations

• How can we apply multiple transformations?

\[ A \cdot (B \cdot (C \cdot v)) \]

• Is it the same as?

\[ B \cdot (A \cdot (C \cdot v)) \]
Transformations

- In some graphics libraries you assign the **position**, **rotation**, **translation** and possibly the **scale** individually.
Transformations

- In some graphics libraries you assign the position, rotation, translation and possibly the scale individually.

```javascript
object.position.set(2.7, 1.2, 0);
object.scale.set(2.4, 0.1, 0.4);
object.rotation.set(0, toRad(180), 0);
```
Transformations

- In some graphics libraries you assign the position, rotation, translation and possibly the scale individually.

- To the GPU the transformations are sent as a matrix (model matrix).
Transformations

• In some graphics libraries you assign the **position**, **rotation**, **translation** and possibly the **scale** individually.

• To the GPU the transformations are sent as a matrix (**model matrix**).

\[
\text{projectionMatrix} \cdot \text{viewMatrix} \cdot \text{modelMatrix} \cdot v
\]

\[
P \cdot V \cdot M \cdot v
\]
Transformations

• In some graphics libraries you assign the position, rotation, translation and possibly the scale individually.

• To the GPU the transformations are sent as a matrix (model matrix).

• Questions about transformations?
Scene Graph

- Dependency between (parts of) objects.
Scene Graph

Head
$S \cdot H \cdot \mathbf{v}$

Body
$S \cdot B \cdot \mathbf{v}$

Left hand
$S \cdot B \cdot L \cdot \mathbf{v}$

Right hand
$S \cdot B \cdot R \cdot \mathbf{v}$
Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
Matrix Stack

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• *Current state* is in the **top of the stack**

  1) $M = \text{Identity}$, push($M$)
Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
- *Current state* is in the **top of the stack**

1) \(M = \text{Identity}, \text{push}(M)\)
2) \(M *= S, \text{push}(M)\)  
   Move to snowman's space
Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
- *Current state* is in the **top of the stack**

1) \(M = \text{Identity}, \text{push}(M)\)
2) \(M *= S, \text{push}(M)\)
3) \(M *= H, \text{push}(M)\)  \text{Move to head's space}
Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
- *Current state* is in the **top of the stack**

1) $M = \text{Identity}, \ \text{push}(M)$
2) $M *= S, \ \text{push}(M)$
3) $M *= H, \ \text{push}(M)$
4) *Draw head vertices*
Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
- **Current state** is in the **top of the stack**

1) \( M = \text{Identity}, \) push(M)
2) \( M *= S, \) push(M)
3) \( M *= H, \) push(M)
4) *Draw head vertices*

We now want to get back to the snowman's space
Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
- *Current state* is in the **top of the stack**

1) \( M = \text{Identity}, \) push(\( M \))
2) \( M *= S, \) push(\( M \))
3) \( M *= H, \) push(\( M \))
4) *Draw head vertices*
5) pop(), \( M = \text{top()} \)
Matrix Stack

• Stack can be used to save and load matrices (intermediary states)

• *Current state* is in the **top of the stack**

2) …
3) $M *= H$, push($M$)
4) *Draw head vertices*
5) pop(), $M = \text{top()}$
Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
- *Current state* is in the **top of the stack**

2) ...
3) M *= H, push(M)
4) *Draw head vertices*
5) pop(), M = top()
6) M *= B, push(M)  
   Move to body's space
Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
- **Current state** is in the **top of the stack**

2) …
3) $M \ast= H$, push($M$)
4) *Draw head vertices*
5) pop(), $M = \text{top()}$
6) $M \ast= B$, push($M$)
7) *Draw body vertices*
Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
- *Current state* is in the **top of the stack**

5) ...

6) \( M \ast= B, \) push(\( M \))

7) *Draw body vertices*

8) ... ?
Matrix Stack

- Each (part of an) **object** can be modelled in its own **local space**.
Matrix Stack

- Each (part of an) object can be modelled in its own local space.
- When we traverse the scene graph, important intermediary states are saved / loaded.
Matrix Stack

- Each (part of an) **object** can be modelled in its own **local space**.
- When we traverse the scene graph, important intermediary states can saved / loaded.
- No need to recalculate same matrix multiplications many times or find inverse transformations.

\[
M = A \cdot B \cdot D \cdot D^{-1} = A \cdot B
\]

vs

```
stack.pop(), \quad M = \text{stack.top()}
```
Matrix Stack

- Each (part of an) **object** can be modelled in its own **local space**.
- When we traverse the scene graph, important intermediary states can saved / loaded.
- No need to recalculate same matrix multiplications many times or find inverse transformations.
- Questions about the matrix stack?
What new did you find out today?

What more would you like to know?

Next time

Frames of reference; projections