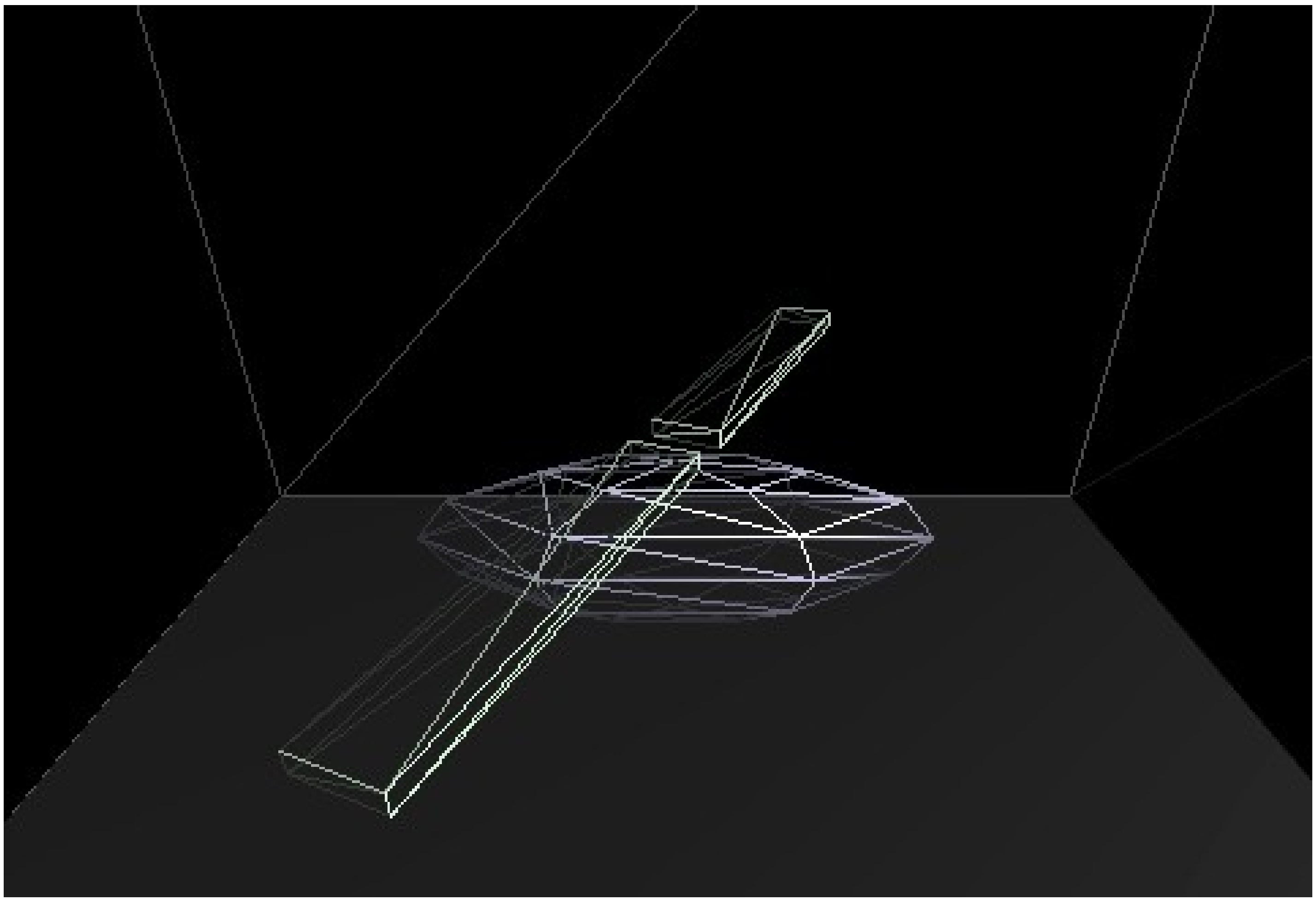
Computer Graphics

MTAT.03.015

Raimond Tunnel

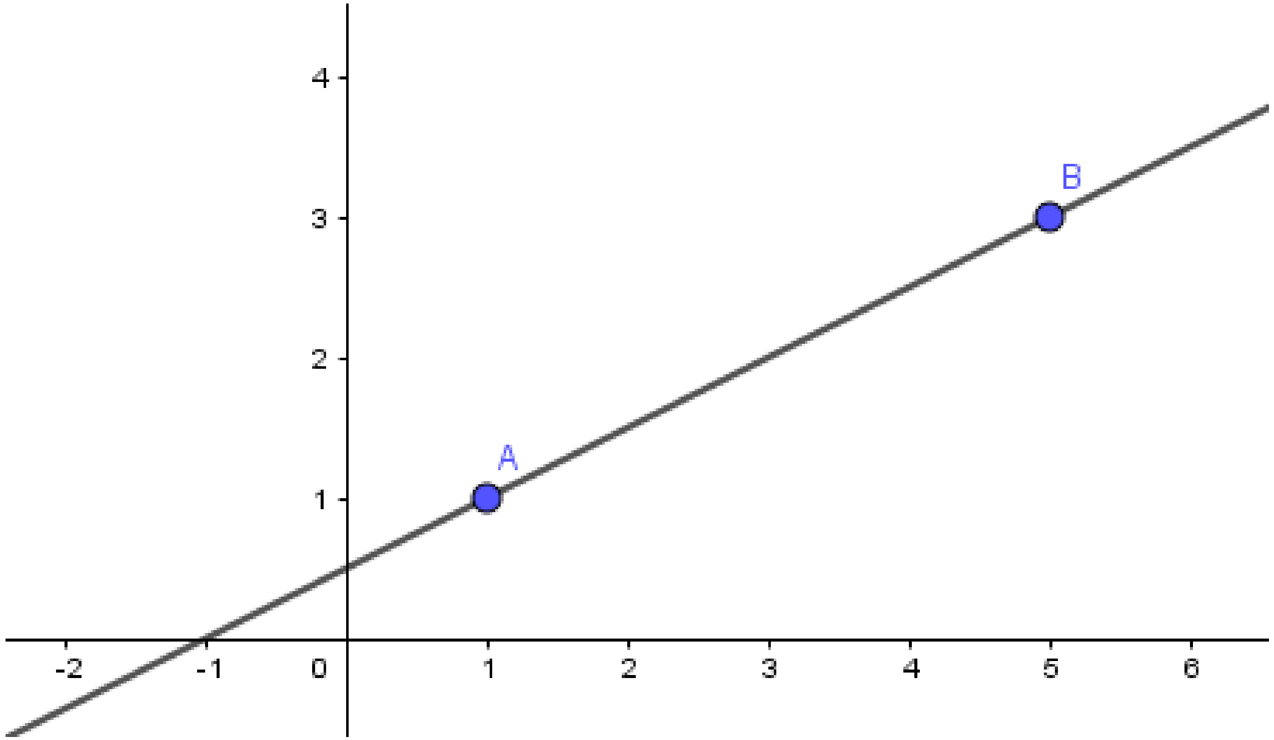
Study IT in .ee

The Road So Far...



Curves

- Line interpolates between 2 points.



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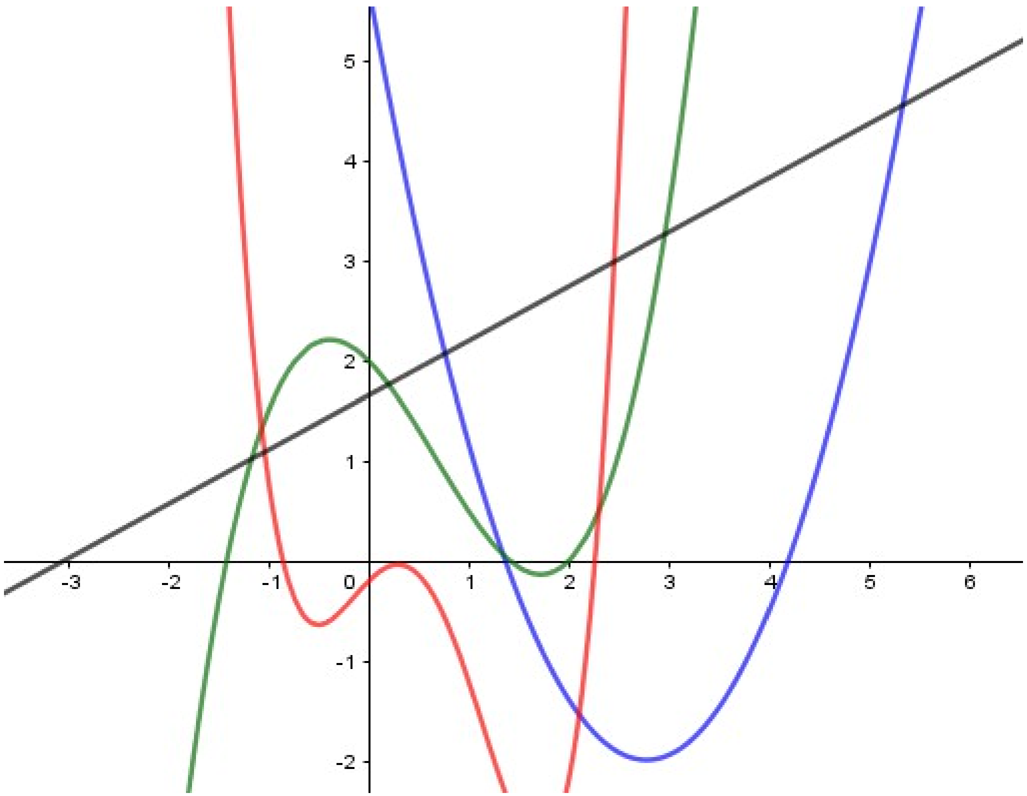
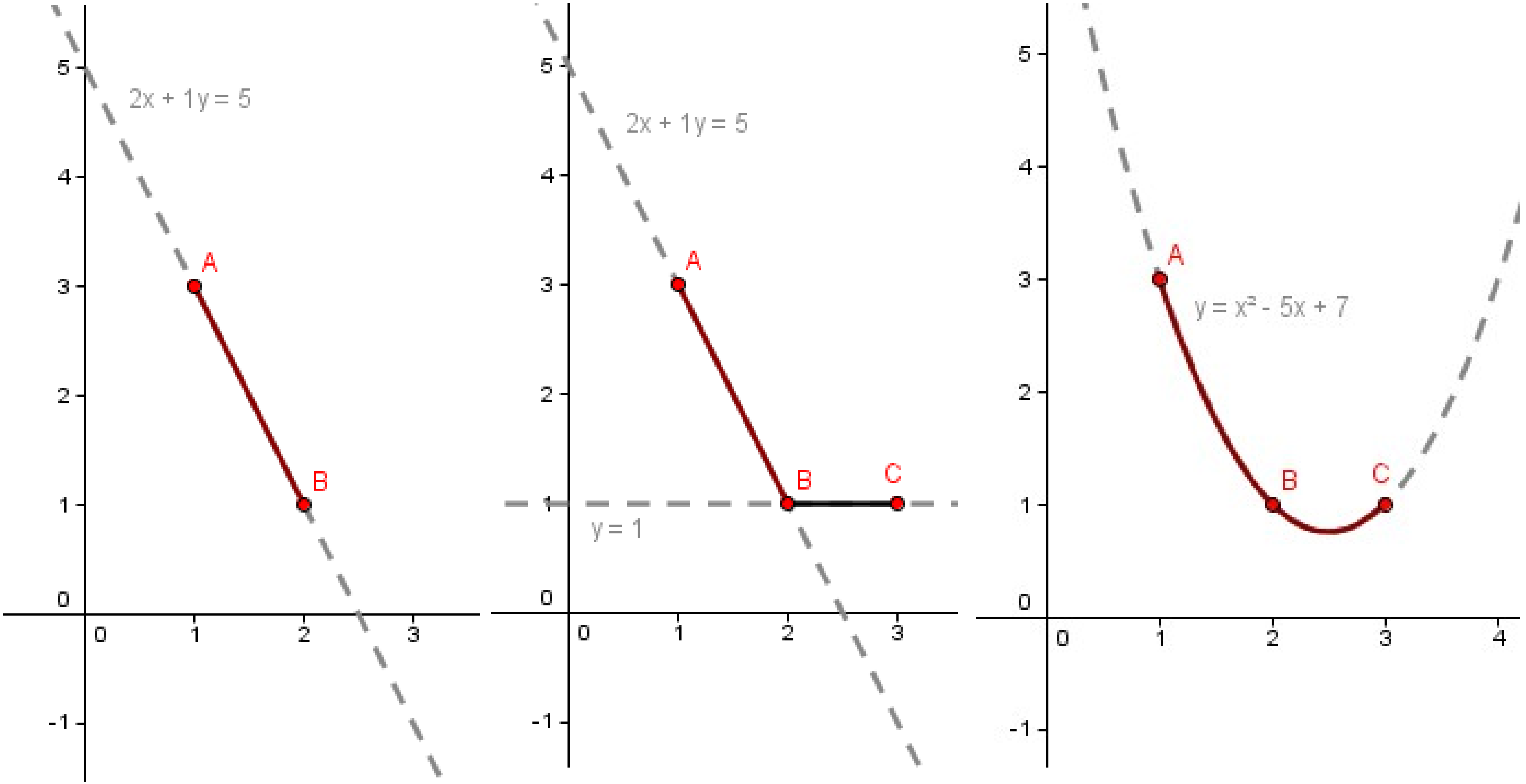


Curves

- Line interpolates between 2 points.
- Mathematically there are higher polynomials to interpolate between more points
- How many points you need, to construct a n-th degree polynomial through it?

Curves



Curves

- Constructing a parabola through 3 points:

$$f(x) = a_2 \cdot x^2 + a_1 \cdot x + a_0$$

Just like we found a line equation in the first lectures

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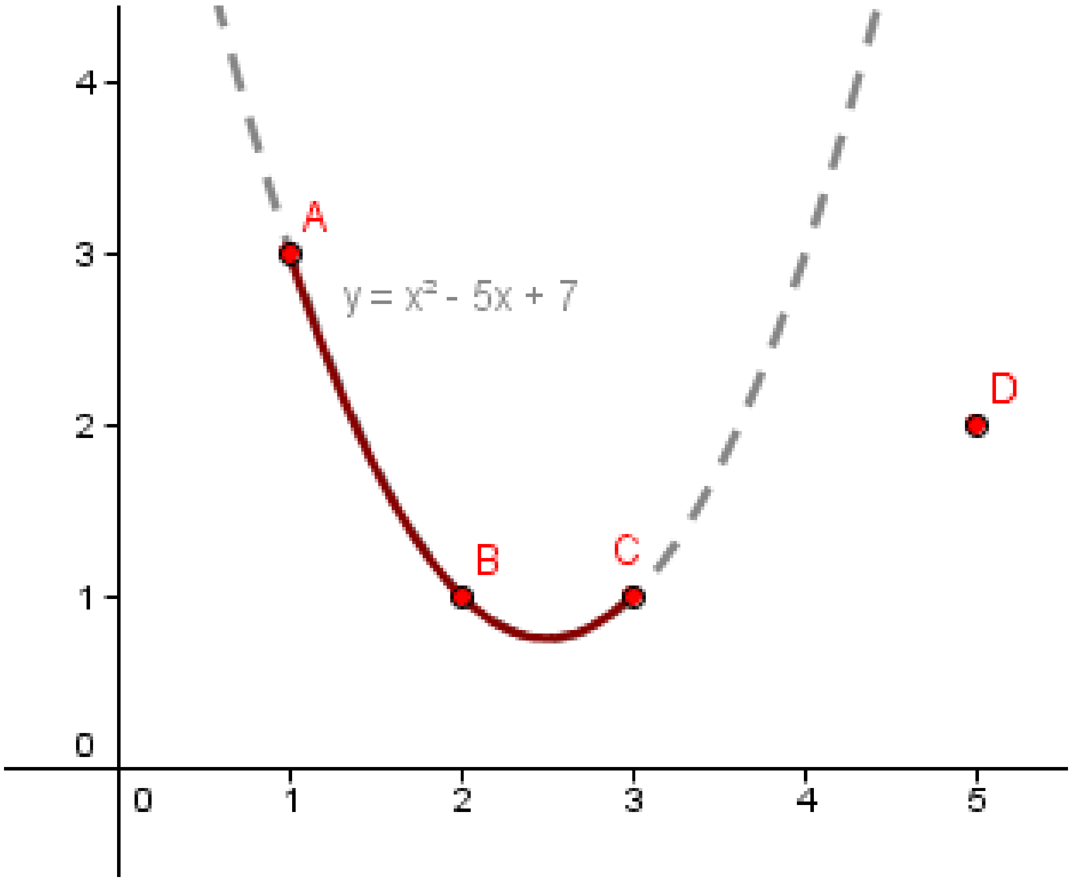
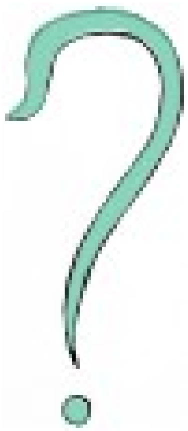
- We know: $f(1) = 3,\ \ f(2) = 1,\ \ f(3) = 1$
- **3 unknowns, 3 constraints, we can solve it.**
- http://www.wolframalpha.com/input/?i=a*1+%2B+b*1+%2B+c+%3D+3%2C+a*4+%2B+b*2+%2B+c+%3D+1%2C+a*9+%2B+b*3+%2B+c+%3D+1

Curves

- What choices we have with 4 points?





One additional point meant another line, could we have 2 parabolas here?

Curves

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Curves

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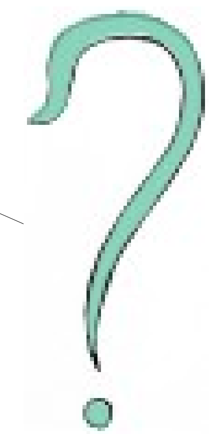
- They can also be on the derivative of it.

$$g(x) = a_2 \cdot x^2 + a_1 \cdot x + a_0 \qquad g'(x) = ?$$



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$$g(5) = 2$$

$$g'(3) = f'(3) = 1$$

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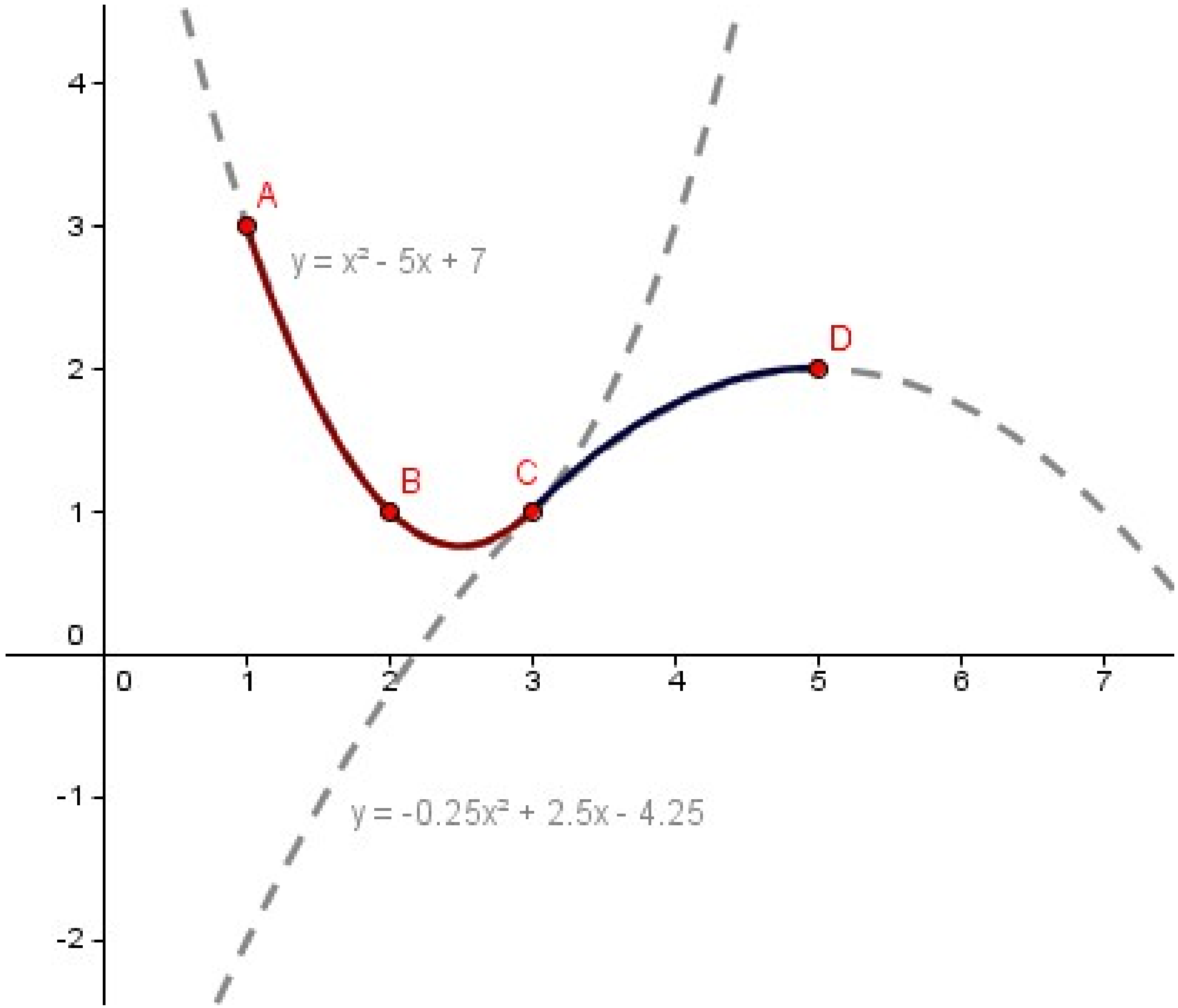
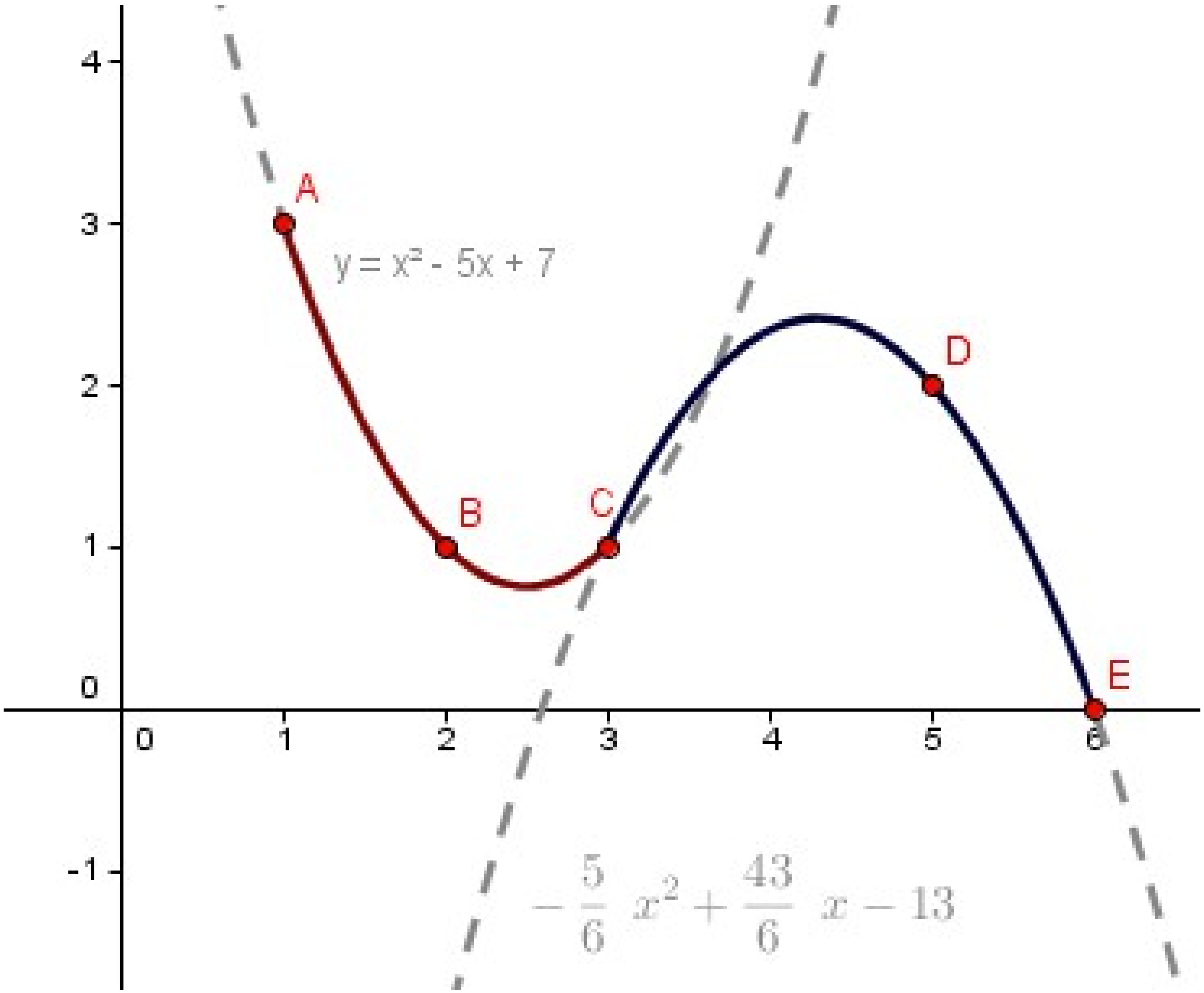


Smoothness

- What if we have 5 points and we put two parabolas through them without accounting for the derivative?

Smoothness

- That spline is not C^1 smooth.



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    And the object is of G^{n-1}

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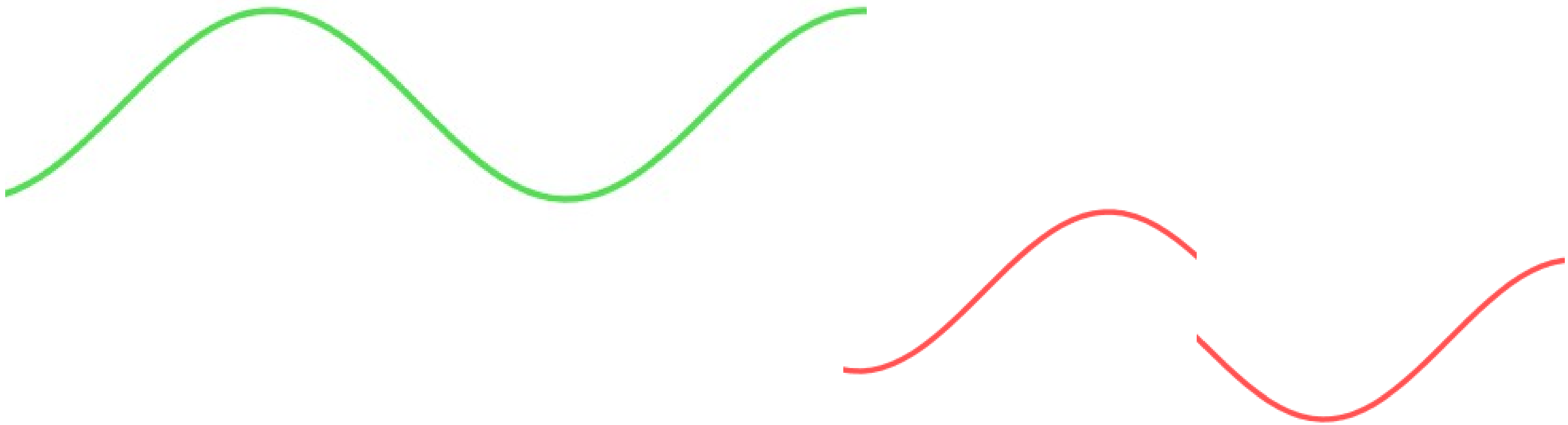
- Different levels of smoothness:
  - C^0 – Curve itself is continuous



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- Different levels of smoothness:
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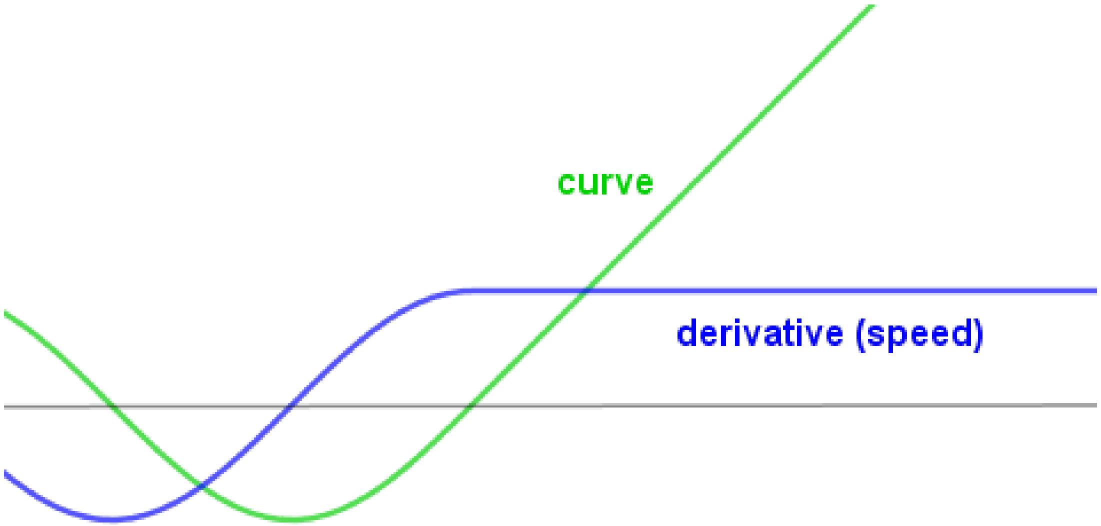

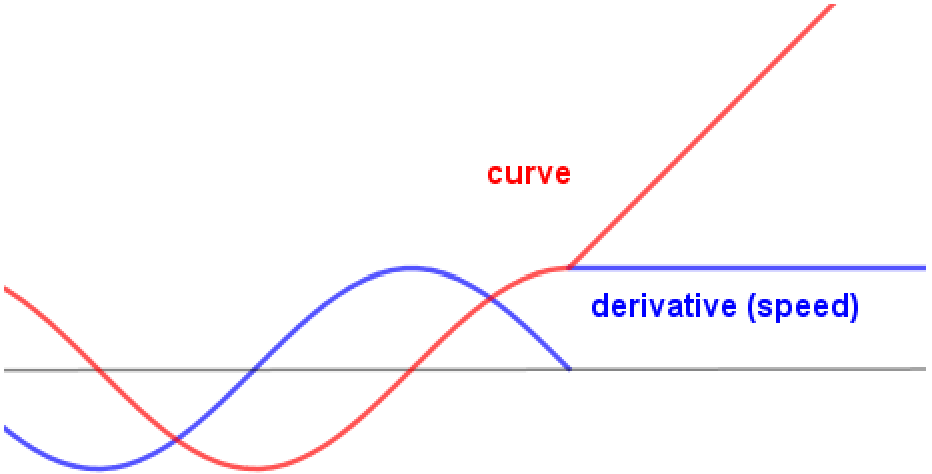

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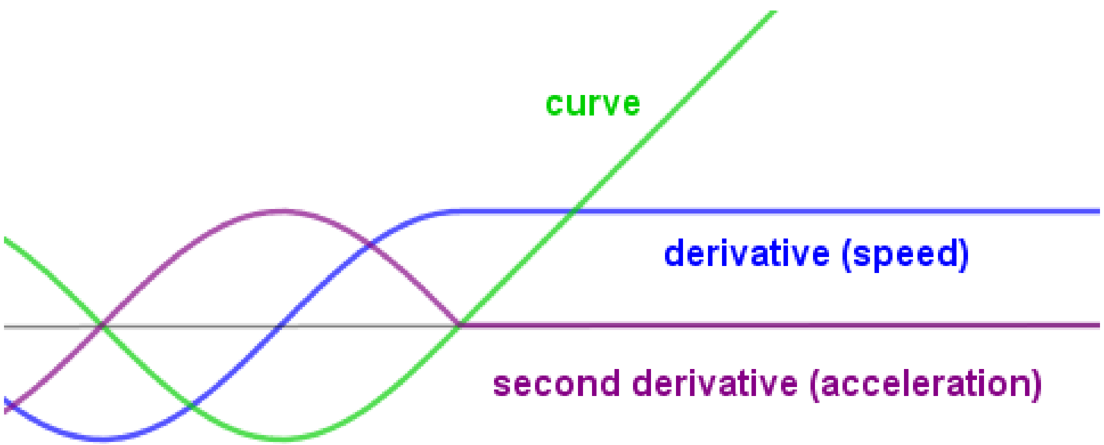

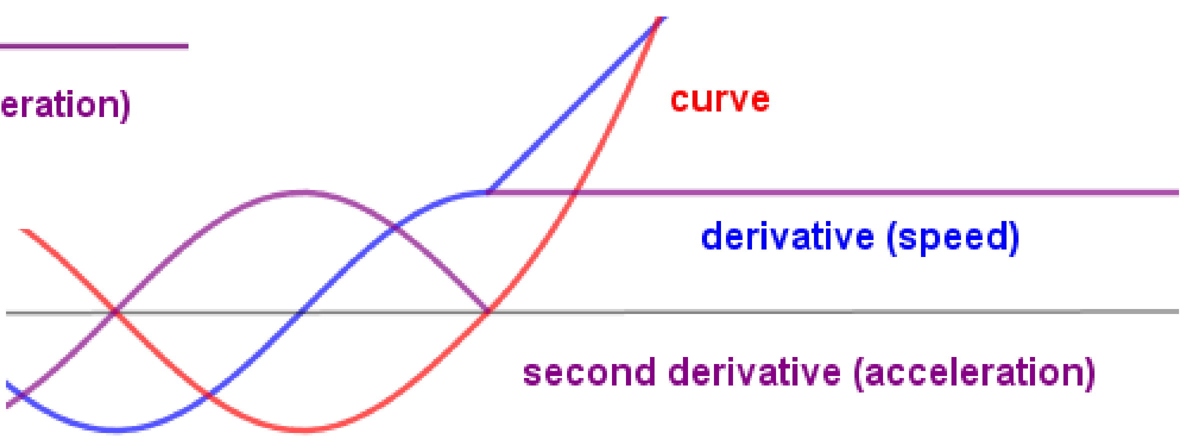

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- Often times C^1 or C^2 smooth curves are enough in computer graphics.
- If we put quadratic curves together, so that the spline is C^1 smooth, how to get C^2 smoothness?

Find the second derivatives of our previous example...

Parametric Curves

- Implicit form: $f(x)=a_2 \cdot x^2 + a_1 \cdot x + a_0$
  - Good for testing points in a curve
  - Finding collisions

For your regular mathematical quadratic fun.

Parametric Curves

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- Parametric form: $g(t) = (t + x_0,\ a_2 \cdot t^2 + y_0) = (x,\ y)$

$$x_0 = \frac{-a_1}{2 \cdot a_2}, \quad y_0 = f(x_0)$$

  - Good for **generating points on the curve**

Parametric form of a quadratic polynomial: http://www.nabla.hr/PC-ParametricEqu2.htm

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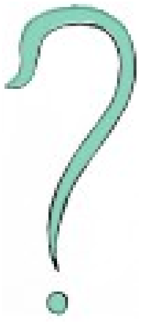
  - Good for generating points on the curve
  - **What other parametric equations you know?**



Parametric Curve Construction

- We want to find the **vector coefficients** a_i for a **function of *t* (time)**, where $t \in [0..1]$.

Quadratic: $$curve(t) = a_0 + a_1 \cdot t + a_2 \cdot t^2$$

Cubic: $$curve(t) = a_0 + a_1 \cdot t + a_2 \cdot t^2 + a_3 \cdot t^3$$

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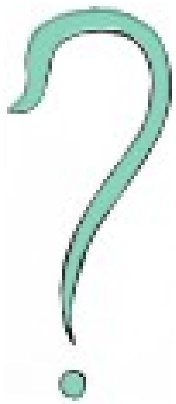

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$$curve(0) = (1,\ 3) = p_0$$

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Control points

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- Usually the system of constraints is written in a **constraint matrix**.

Parametric Curve Construction

- Constraint matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0.25 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}$$

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In short: $C \cdot a = p$

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In our example:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \end{pmatrix}$$

Inverse (and other stuff) matrix calculator: http://www.bluebit.gr/matrix-calculator/calculate.aspx

Parametric Curve Construction

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$$a_0 = b_{0,0}\, p_0 + b_{0,1}\, p_1 + b_{0,2}\, p_2$$
$$a_1 = b_{1,0}\, p_0 + b_{1,1}\, p_1 + b_{1,2}\, p_2$$
$$a_2 = b_{2,0}\, p_0 + b_{2,1}\, p_1 + b_{2,2}\, p_2$$

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Parametric Curve Construction

- Let us look at the entire curve:

$$
\begin{aligned}
& curve(t) = \\
& \quad = b_{0,0}\, p_0 + b_{0,1}\, p_1 + b_{0,2}\, p_2 + \\
& \quad + b_{1,0}\, p_0 t + b_{1,1}\, p_1 t + b_{1,2}\, p_2 t + \\
& \quad + b_{2,0}\, p_0 t^2 + b_{2,1}\, p_1 t^2 + b_{2,2}\, p_2 t^2
\end{aligned}
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$$curve(t)=b_0(t)\cdot p_0+b_1(t)\cdot p_1+b_2(t)\cdot p_2$$

$$b_i(t)=b_{0,i}+b_{1,i}\cdot t+b_{2,i}\cdot t^2$$

Coefficients from one column of the matrix B.

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$$b_i(t) = b_{0,i} + b_{1,i} \cdot t + b_{2,i} \cdot t^2$$

The functions b_i are called **basis / blending functions**.

Parametric Curve Construction

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1) Pick a degree of the curve
2) Fix the parameters (*incl* control points)
3) Create the constraint matrix C
4) Find the basis matrix $B = C^{-1}$
5) Read the blending functions from the basis matrix

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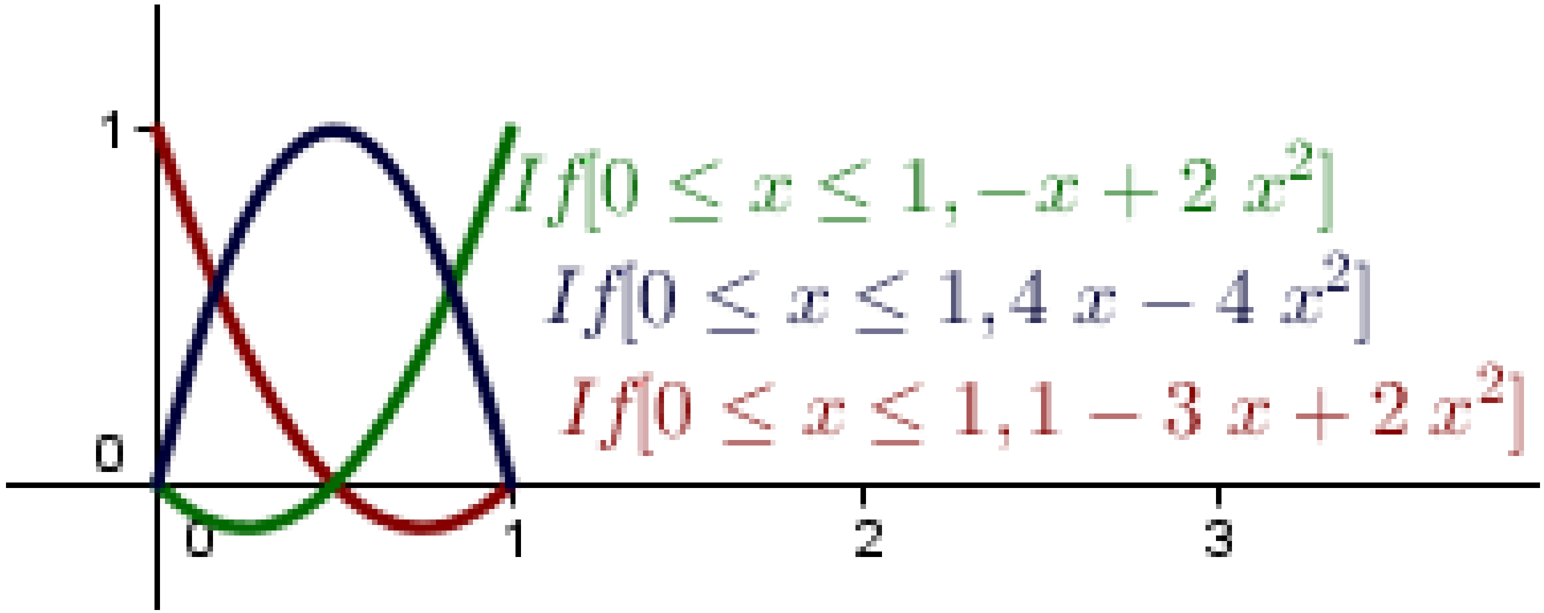

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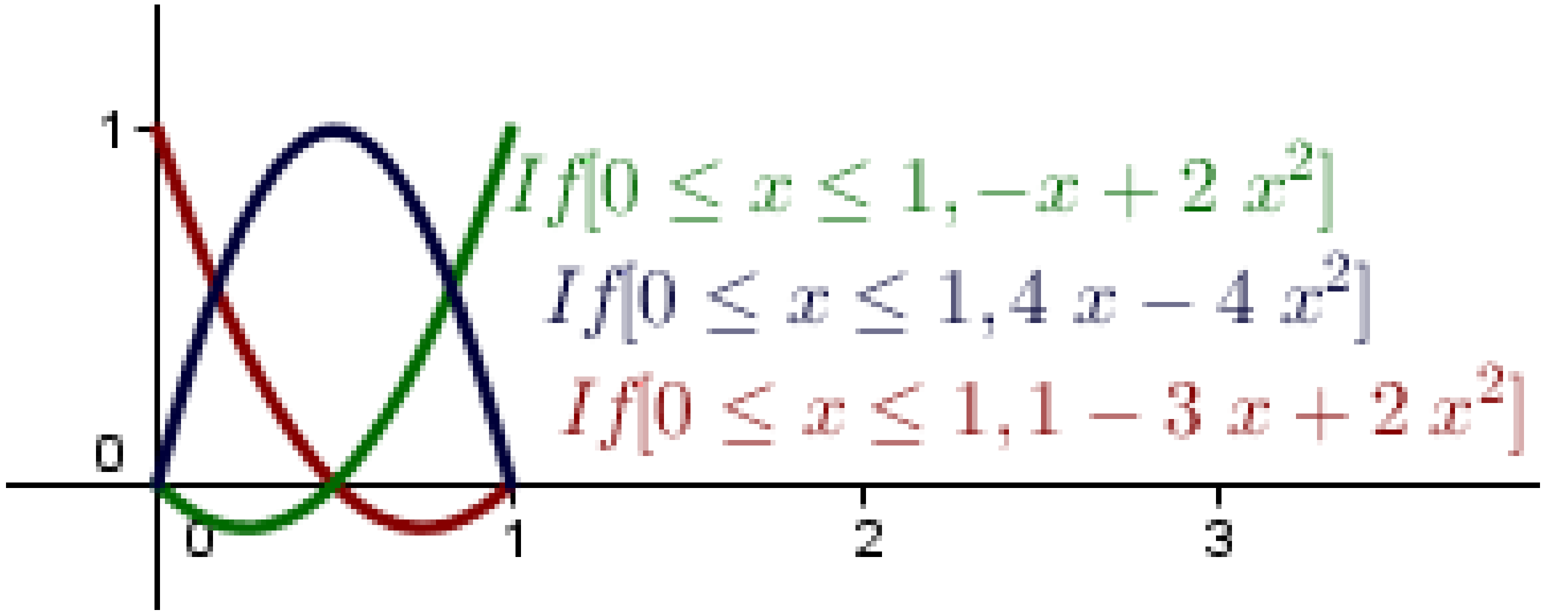

- Different constraints give different functions

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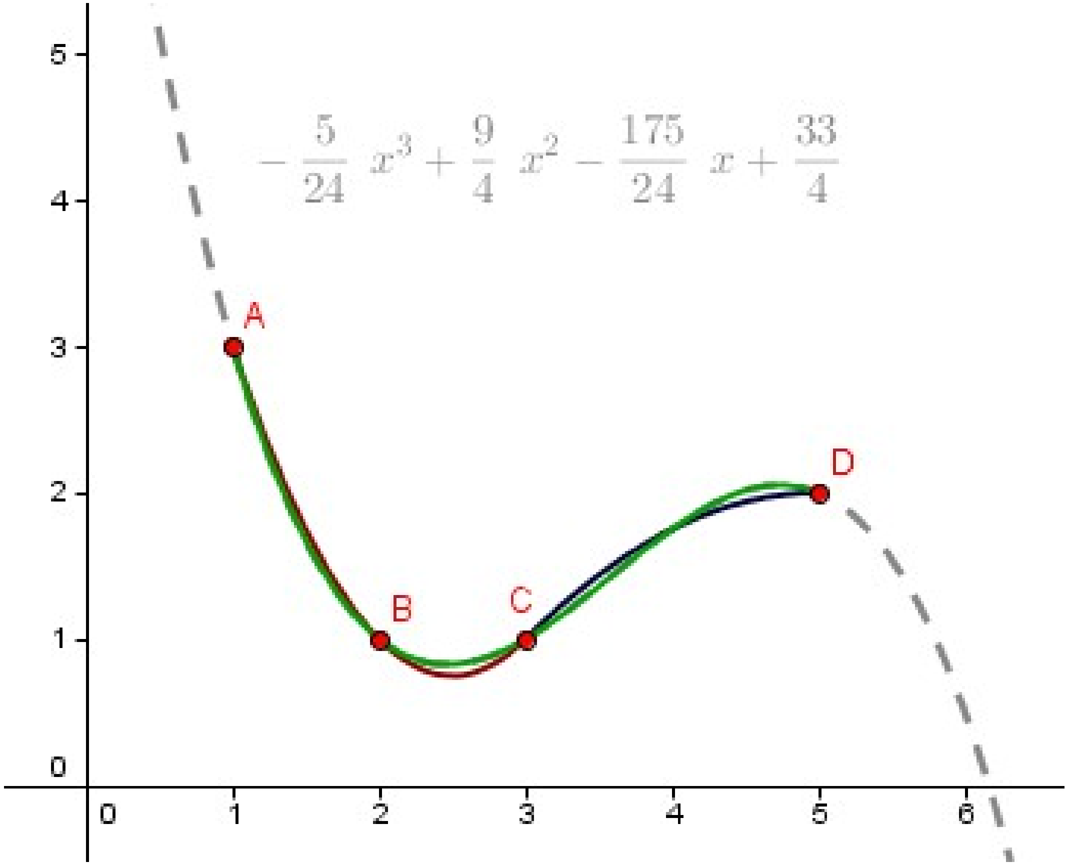
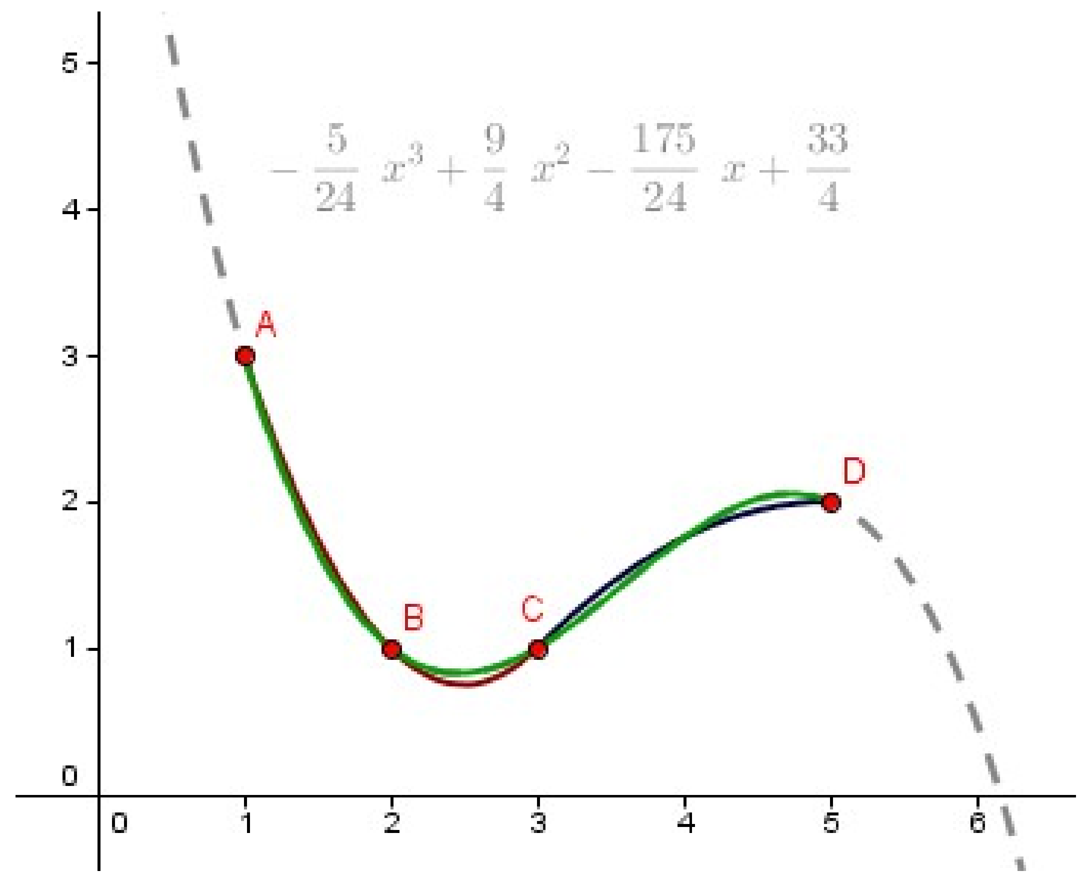


Cubic not Quadratic

- In computer graphics, we usually want to use cubic polynomials, not quadratics.
- Cubic polynomials provide us with 4 possible constraints.
- Splines can achieve C^2 smoothness.



Hermite Spline

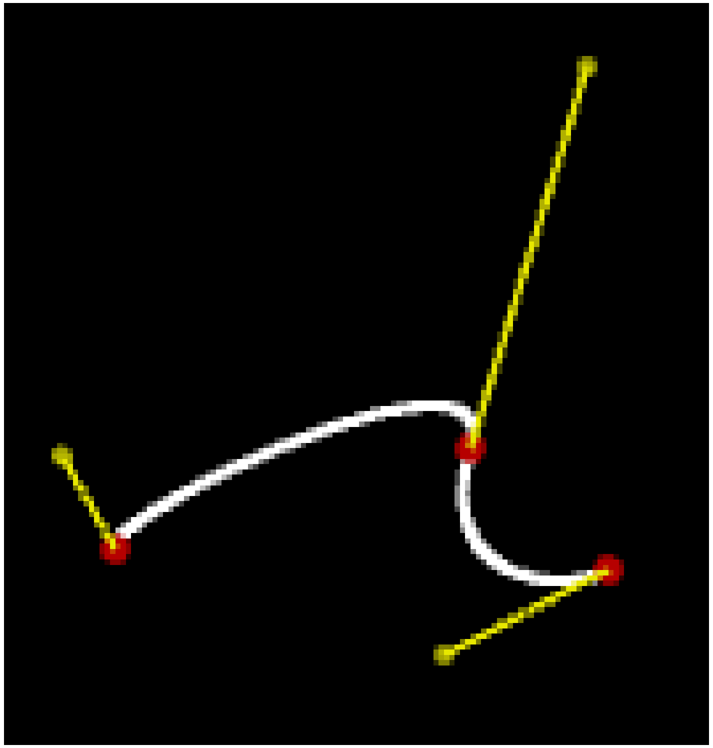
- The derivatives at the endpoints are parameters.
- Segments share the endpoints and derivatives.

$$curve(0) = p_0$$

$$curve'(0) = p_1$$

$$curve(1) = p_2$$

$$curve'(1) = p_3$$



Catmull-Rom Spline

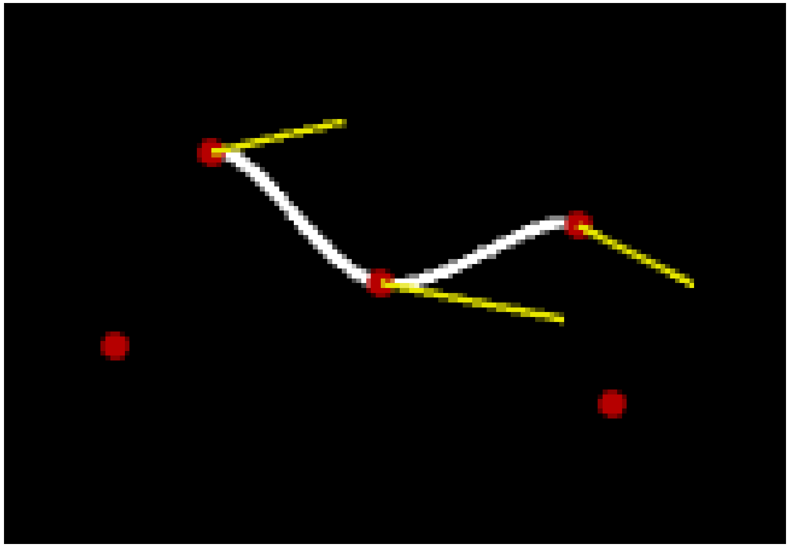
- We interpolate the p_1 and p_2.
- Dervatives are calculated using the other points.

$$curve'(0) = 0.5 \cdot (p_2 - p_0)$$

$$curve(0) = p_1$$

$$curve(1) = p_2$$

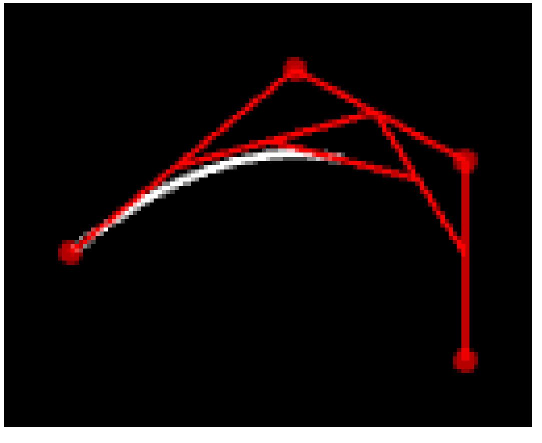
$$curve'(1) = 0.5 \cdot (p_3 - p_1)$$

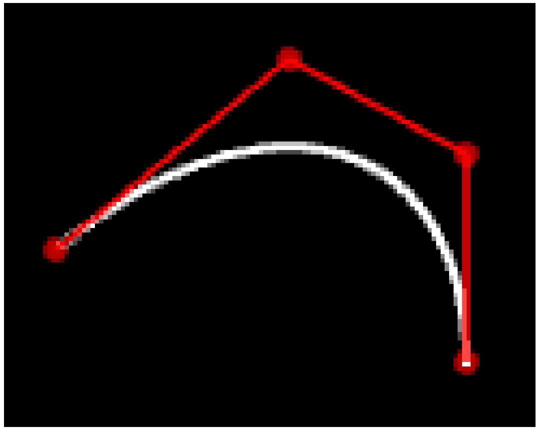


- Only specify start and end derivatives, others are calculated.

Bezier Curve

- Could be constructed using the constraints and finding the blending functions.
- Could also be constructed in a procedural way:
  - Subdivide the lines connecting the control points, into proportions t and $(1-t)$.
  - Do it recusively until at last subdivision, which will give a point on the curve.





Bezier Curve

- That procedure is called De Casteljau's algorithm.
- The corresponding blending functions are called Bernstein polynomials.

Those you already used in the practice session.

$$b_{0,0}(t)=1$$

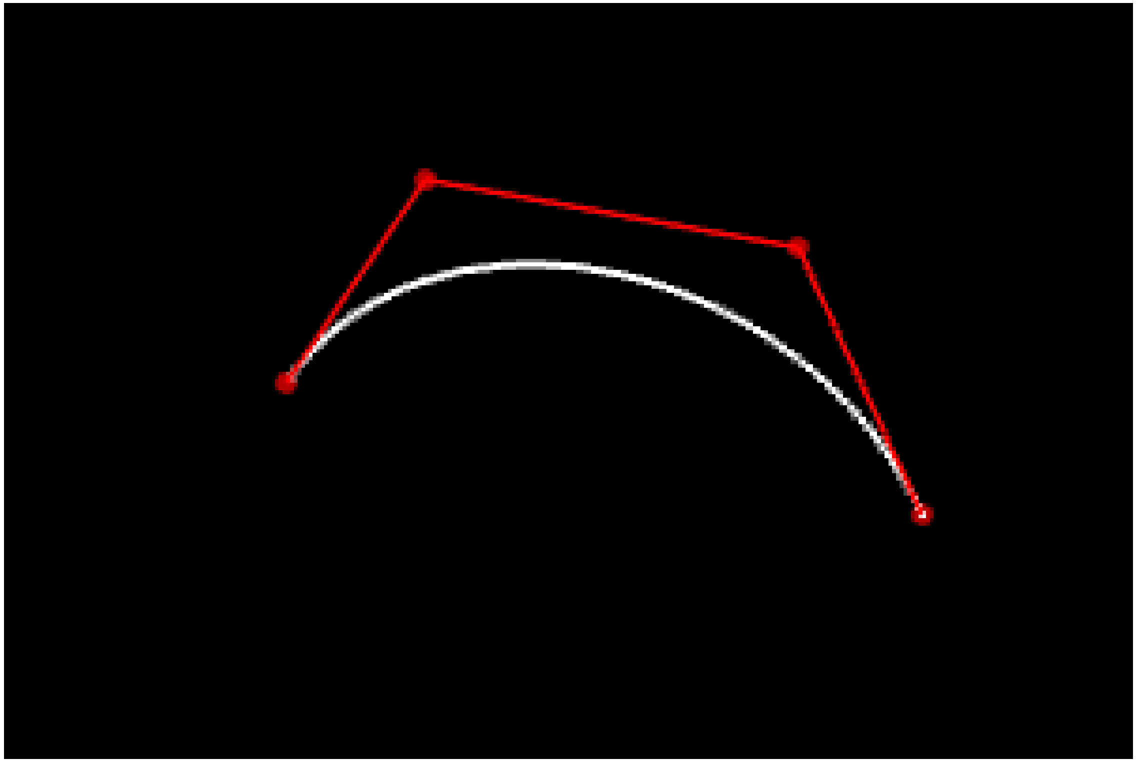
$$b_{0,1}(t)=1-t\,,\quad b_{1,1}(t)=t$$

$$b_{0,2}(t)=(1-t)^2\,,\quad b_{1,2}(t)=2\cdot t\cdot(1-t)\,,\quad b_{2,2}(t)=t^2$$

$$b_{i\,,\,degree}(t)=\binom{degree}{i}\cdot t^i\cdot(1-t)^{degree-i}$$

Bezier Curve

- Always inside the convex hull of the control points.



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- Very widely used (*eg* font rendering)

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- Hermite and Catmull-Rom – are not C^2 smooth.
- Bezier – does not interpolate the control points.

What did you find exciting today?

What more would you like to know?

Next time

Procedural Generation – *Jaanus Jaggo*