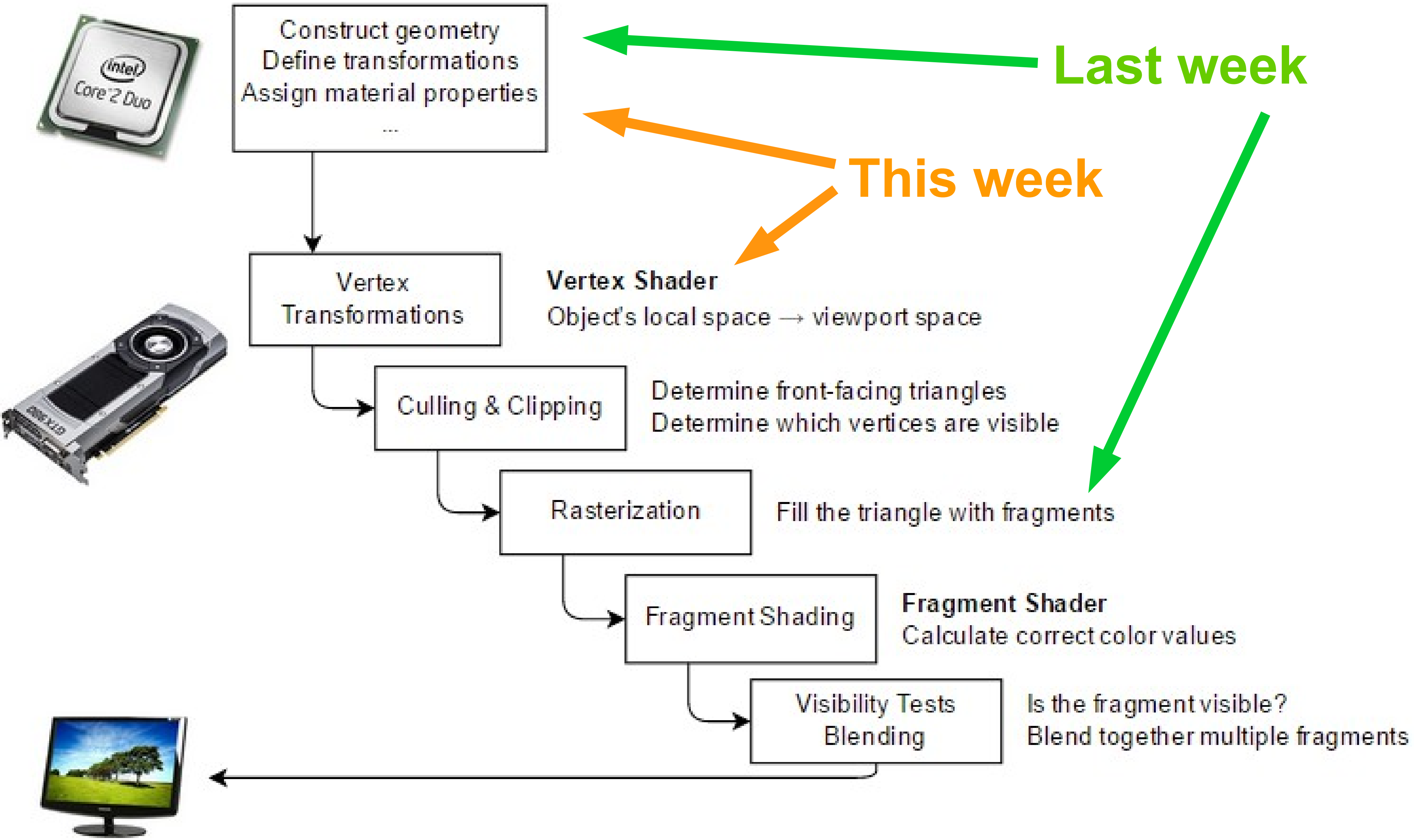
Computer Graphics

MTAT.03.015

Raimond Tunnel

Study IT in .ee

The Road So Far...



Transformations

- Watch the Computerphile video, try to find out:
  1) Why are we using matrices?

The True Power of the Matrix (Transformations in Graphics) – Computerphile
https://www.youtube.com/watch?v=vQ60rFwh2ig

Transformations

- Watch the Computerphile video, try to find out:
  1) Why are we using matrices?
  2) Where do the homogeneous coordinates come in?

The True Power of the Matrix (Transformations in Graphics) – Computerphile
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Linear Transformations

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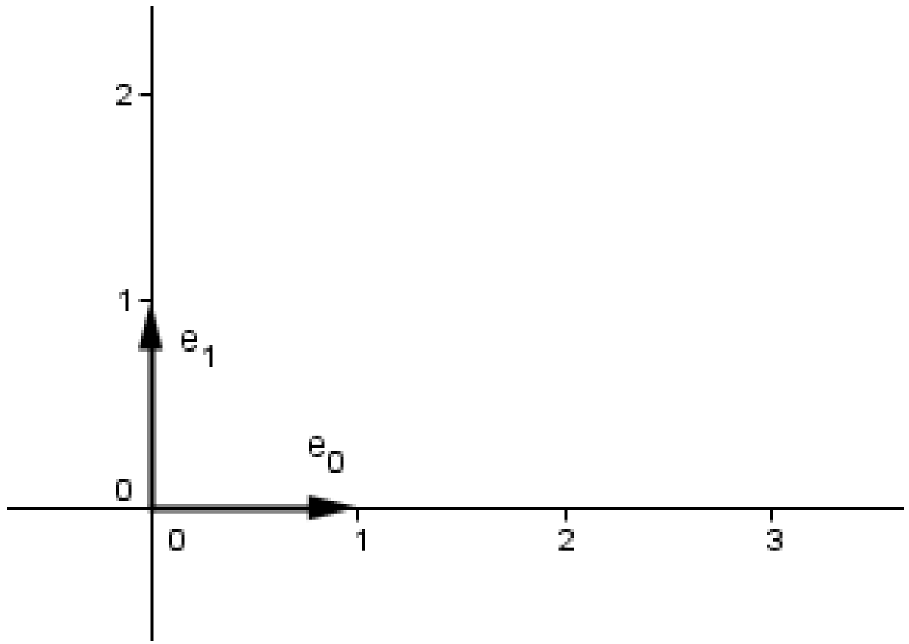
- Satisfies: $f(\alpha \cdot v + \beta \cdot u) = \alpha \cdot f(v) + \beta \cdot f(u)$
- In 3D: $\alpha, \beta \in \mathbb{R} \quad u, v \in \mathbb{R}^3$

Linear Transformations

- Take our vector space of points



Linear Transformations

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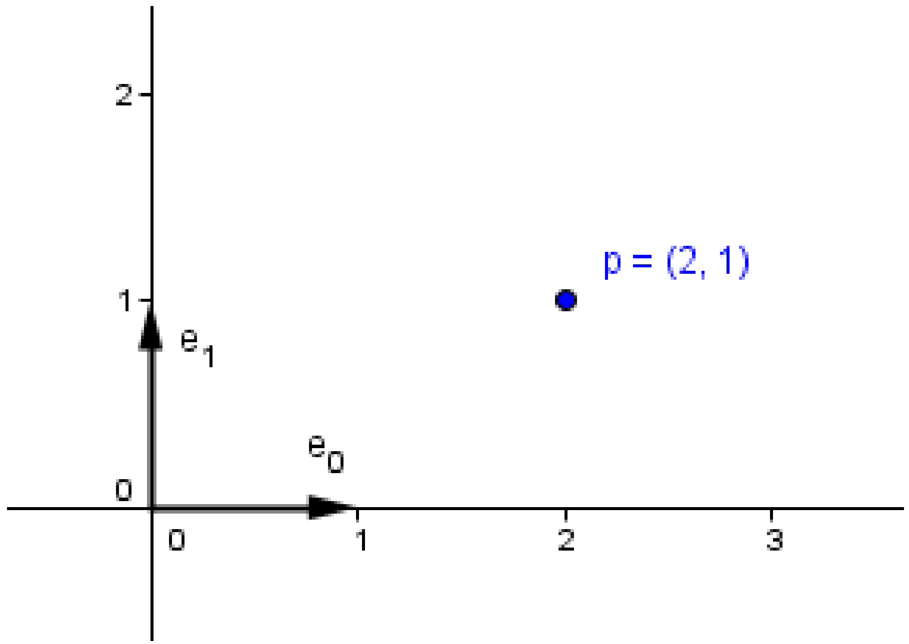
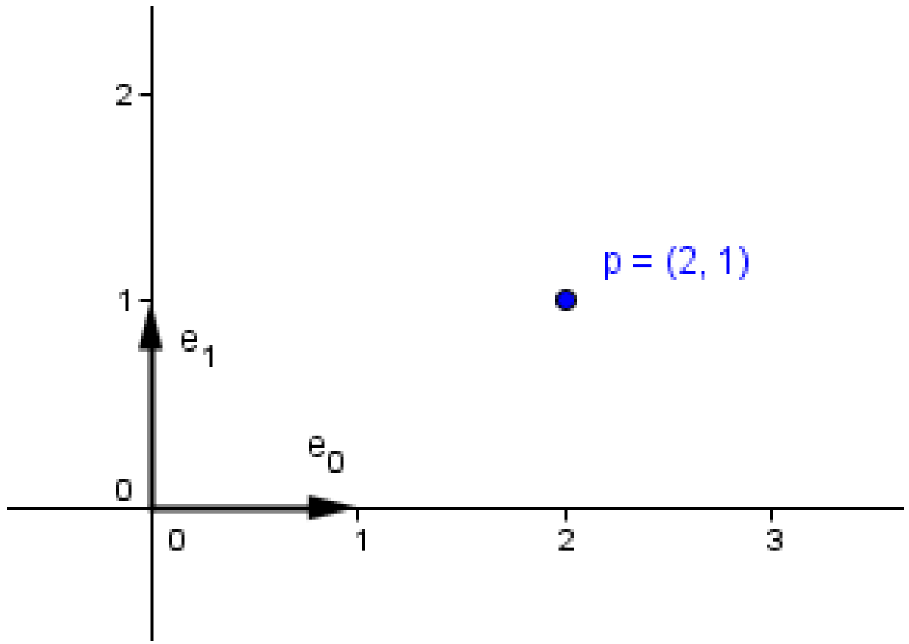
- Take for example a point $p = (2,\ 1)$



Linear Transformations

- Take our vector space of points
- Take for example a point $p = (2, 1)$
- Try mappings:

1) $f(p) = (p_x, \ p_y)$
2) $f(p) = (2 \cdot p_x, \ p_y)$
3) $f(p) = (p_x, \ 2 \cdot p_y)$
4) $f(p) = (2 \cdot p_x, \ 2 \cdot p_y)$



Test the linearity at home...

Linear Transformations

- From Algebra you know that all linear **trans-formations** can be represented **as matrices**.

$$\text{Linear transformation} \rightarrow \text{Matrix}$$

Linear Transformations

- From Algebra you know that all linear **transformations** can be represented **as matrices**.
- **Every matrix** also gives you a **linear transformation**.

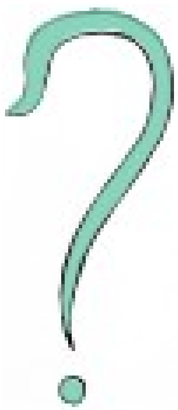
$$\text{Linear transformation} \rightarrow \text{Matrix}$$

$$\text{Linear transformation} \leftarrow \text{Matrix}$$

Linear Transformations

- What would be the matrices for the linear transformations we just saw?

$$f(p) = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$



$$f(p) = (p_x,\ p_y)$$

$$f(p) = (p_x,\ 2 \cdot p_y)$$

$$f(p) = (2 \cdot p_x,\ p_y)$$

$$f(p) = (2 \cdot p_x,\ 2 \cdot p_y)$$

Scale

- Stretches or shrinks the space

2D

$$\begin{pmatrix} a_x & 0 \\ 0 & a_y \end{pmatrix}$$

a_x – x-axis scale factor

a_y – y-axis scale factor

3D

$$\begin{pmatrix} a_x & 0 & 0 \\ 0 & a_y & 0 \\ 0 & 0 & a_z \end{pmatrix}$$

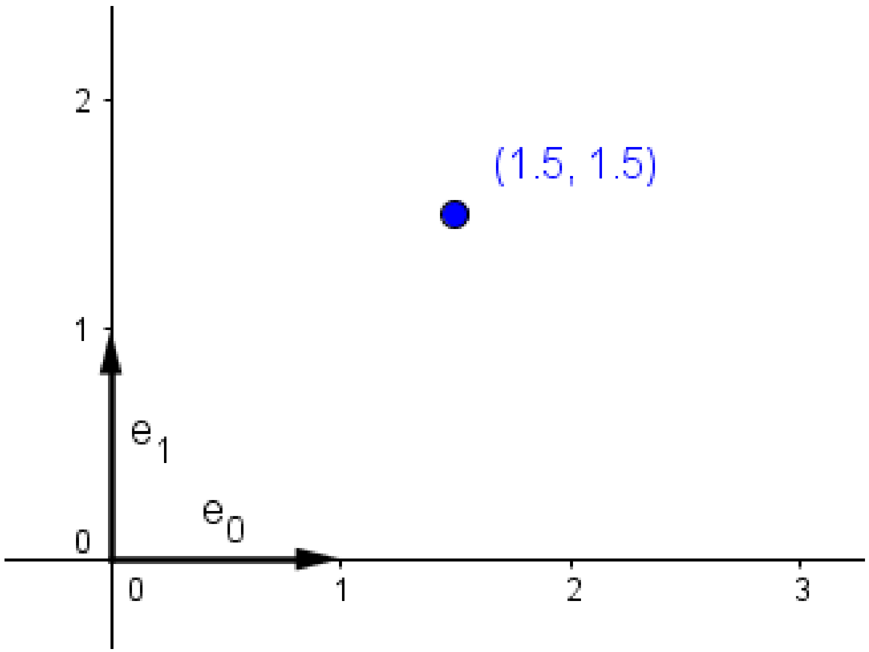
a_x – x-axis scale factor

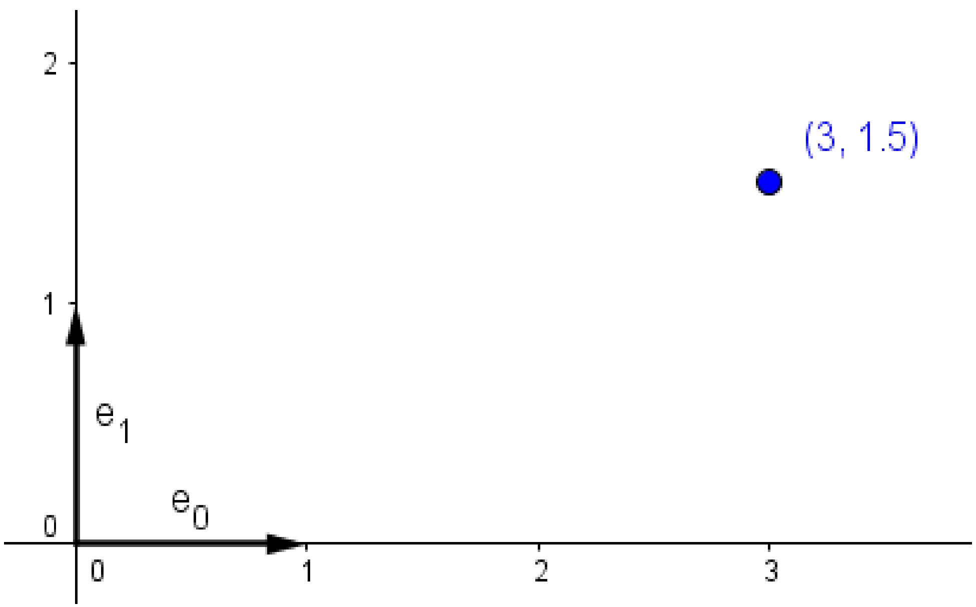
a_y – y-axis scale factor

a_{x} – z-axis scale factor

Scale

- Transformations can be easily understood, if we see what they do with the standard basis





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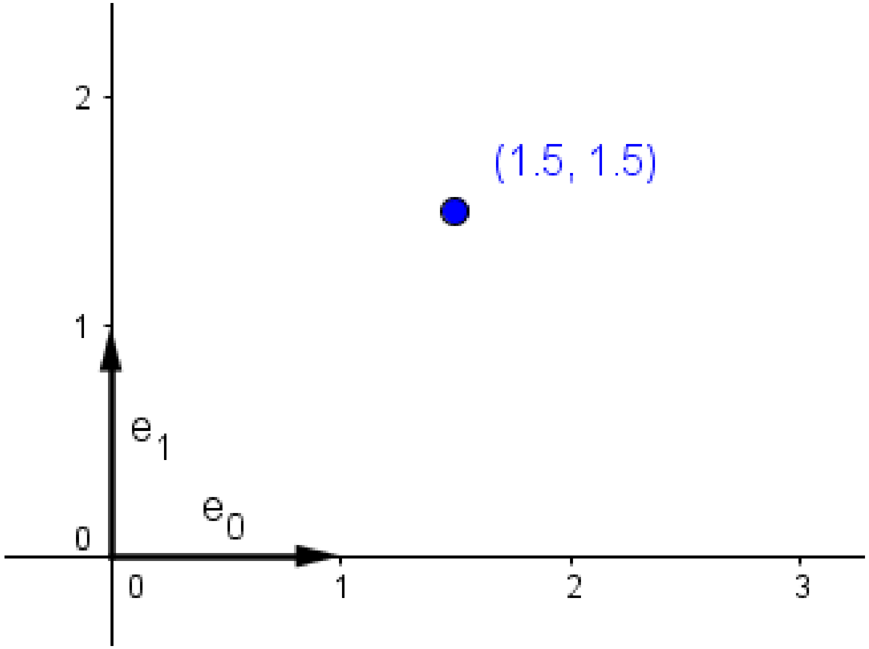

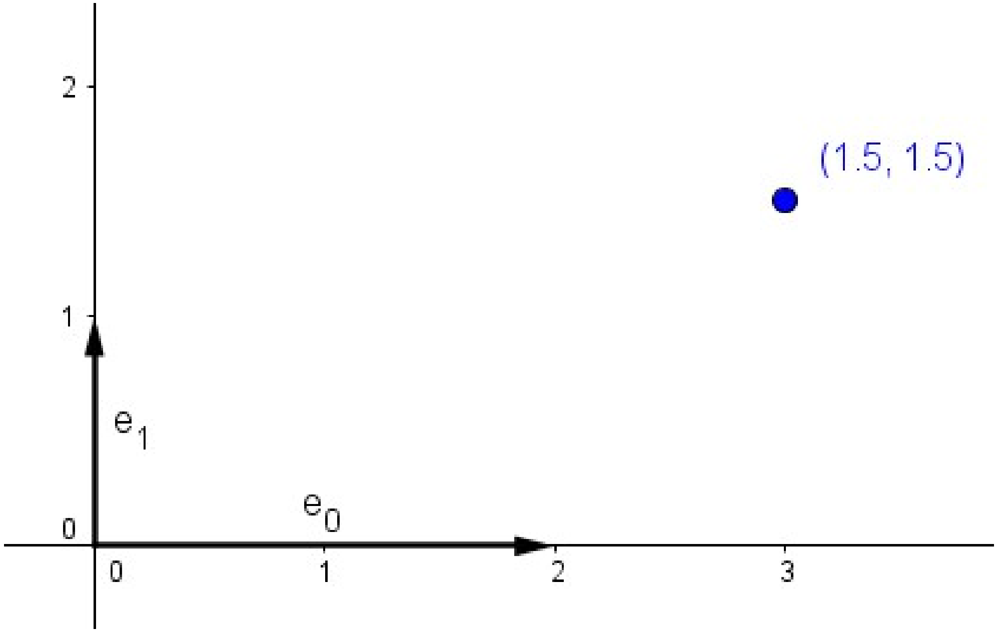

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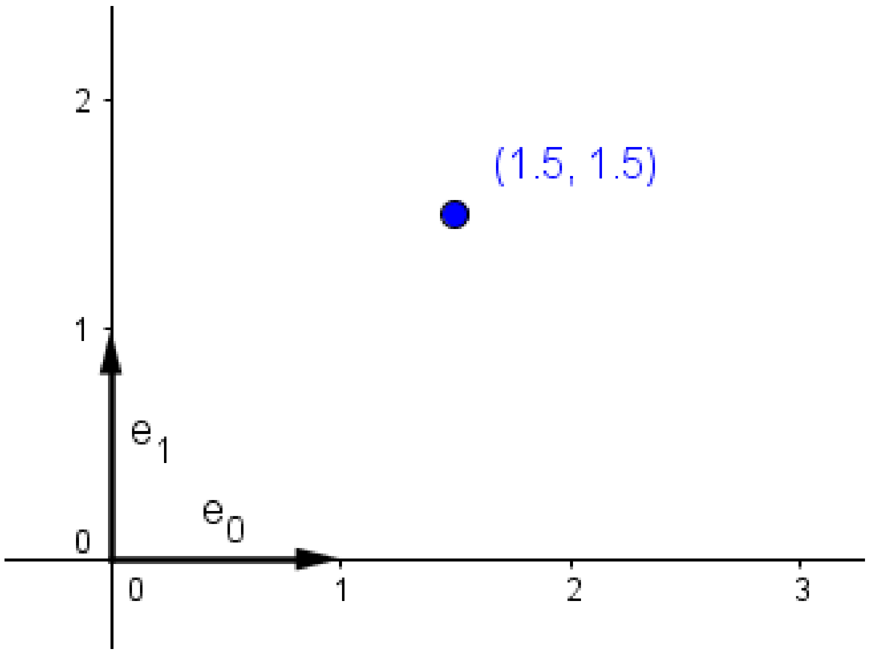
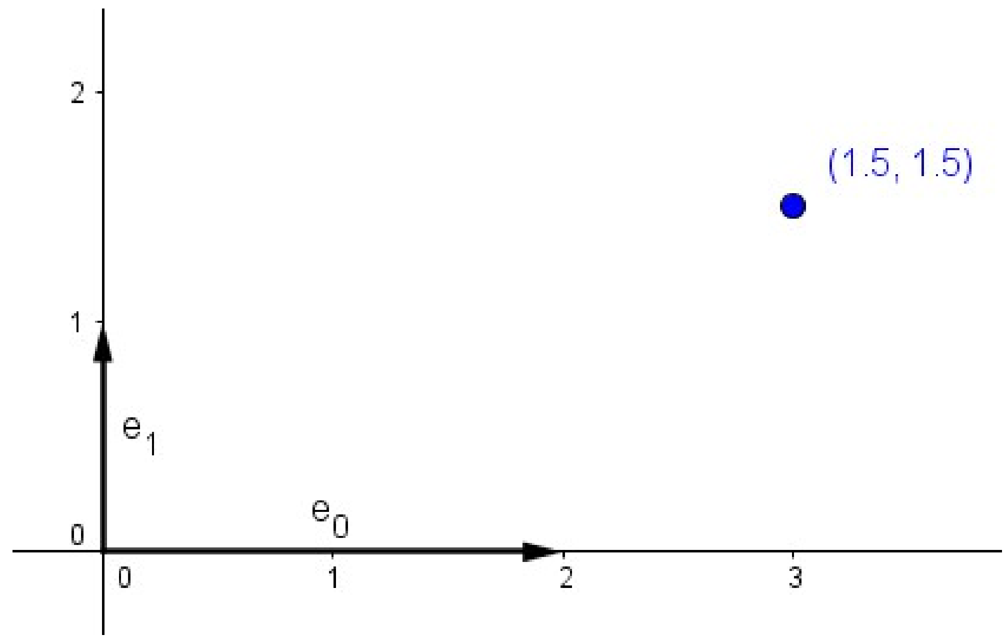
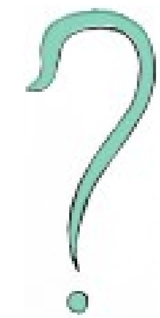




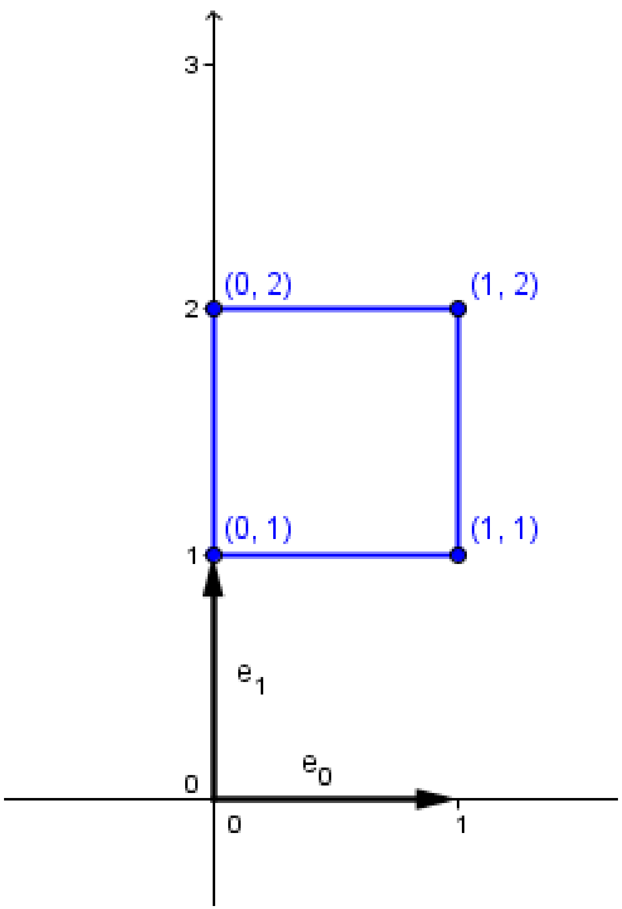
- Furthermore, one can read the transformed standard basis from the columns of the transformation matrix!

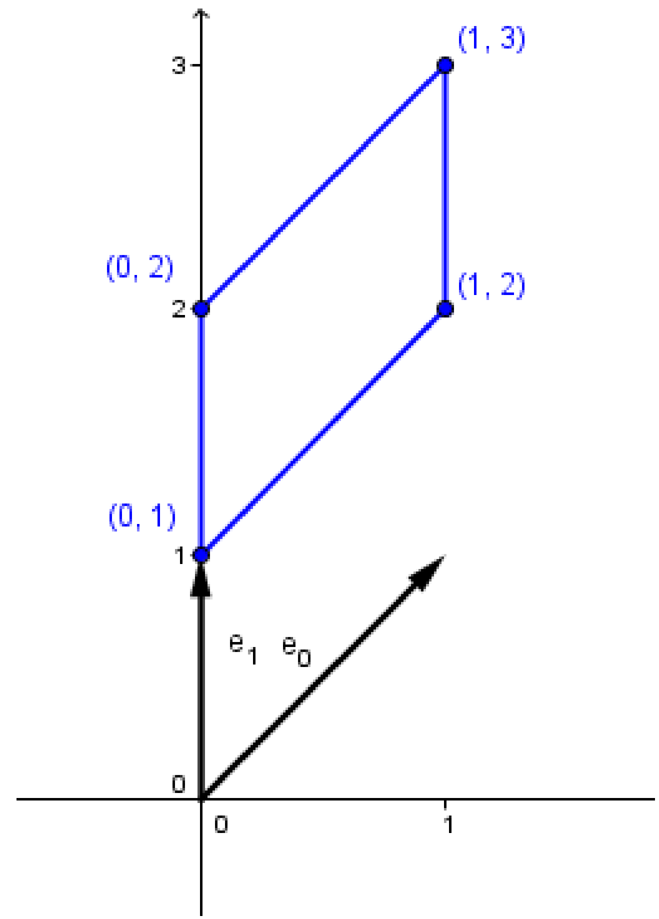
Shear

Shear-x or shear-y?
Matrix?



- Shear-x, shear-y
- Tilts one of the axes parallel to other(s)

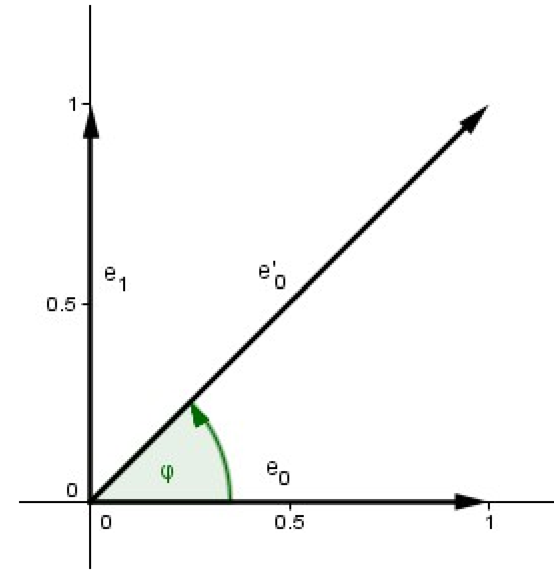




Shear

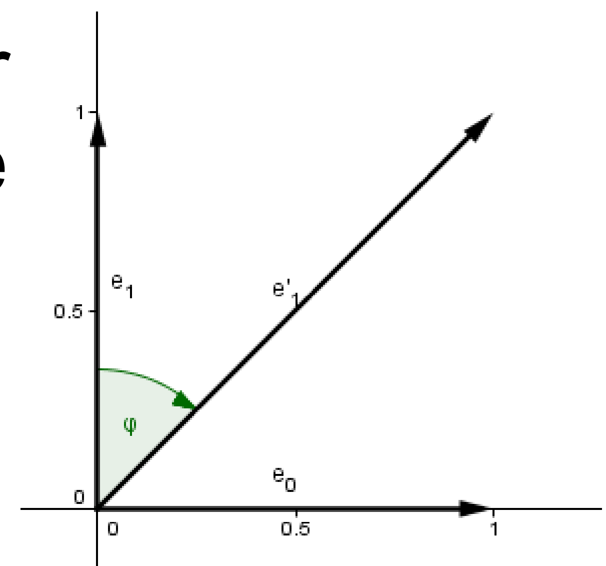
- Shear-y, we tilt the *x* basis vector parallel to *y* by angle φ counter-clockwise

$$\begin{pmatrix} 1 & 0 \\ \tan(\varphi) & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + \tan(\varphi) \cdot x \end{pmatrix}$$



- Shear-x, we tilt the *y* basis vector parallel to *x* by angle φ clockwise

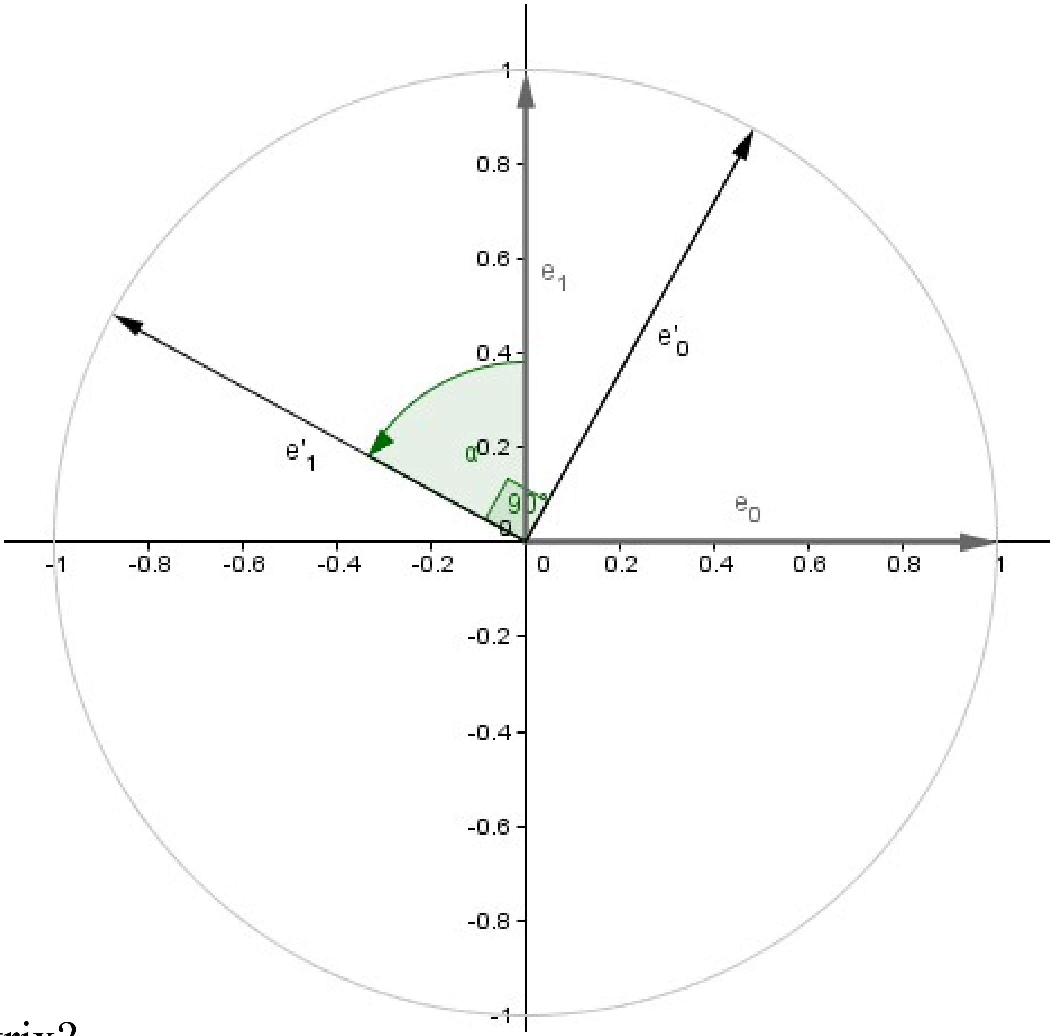
$$\begin{pmatrix} 1 & \tan(\varphi) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \tan(\varphi) \cdot y \\ y \end{pmatrix}$$

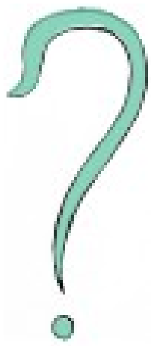


What about in 3D?

Rotation

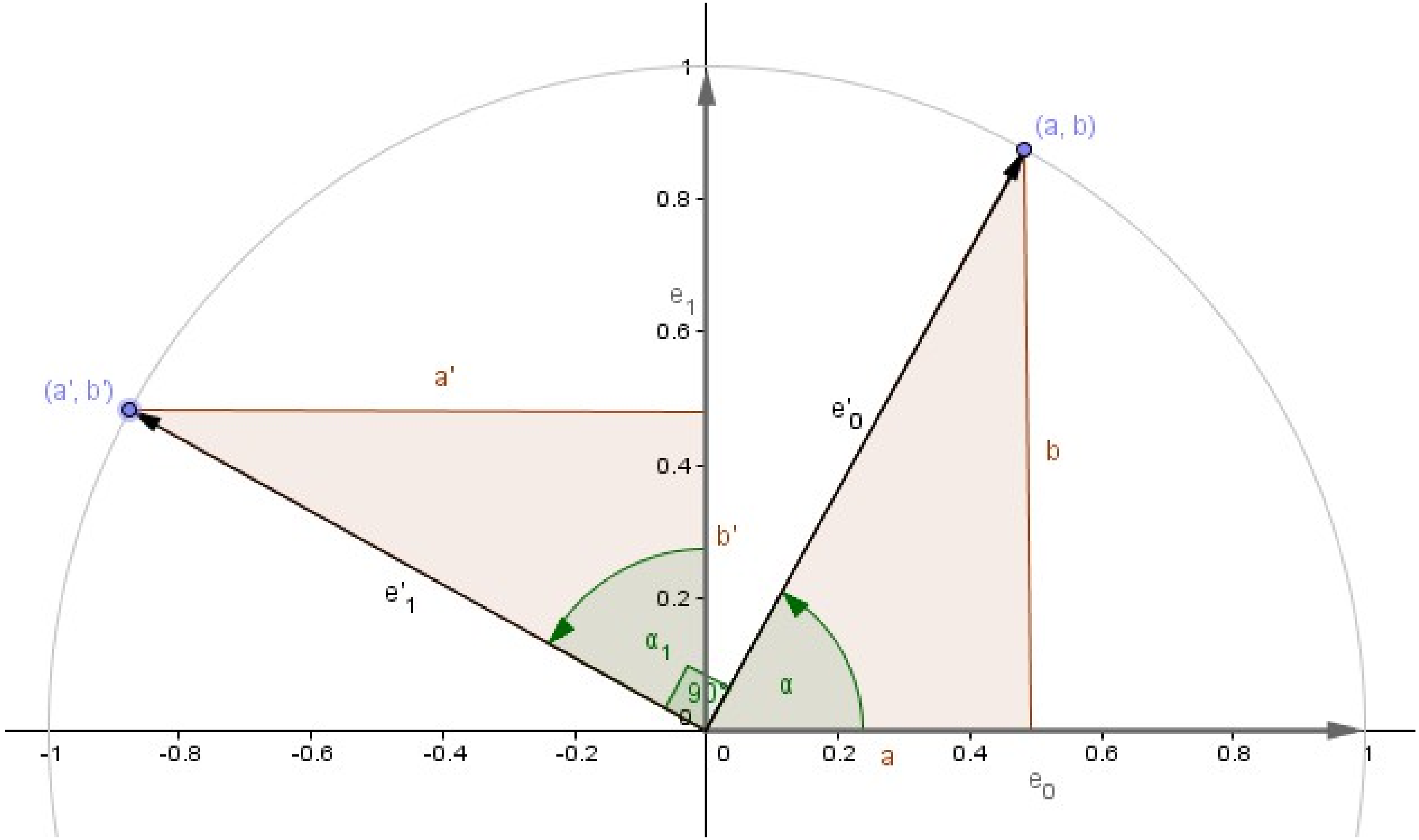
- We want to keep the basis vectors on the unit-circle.





Can you derive the matrix?

Rotation



$$e'_0 = (|a|, |b|) = (\cos(\alpha), \sin(\alpha))$$
$$e'_1 = (|a'|, |b'|) = (-\sin(\alpha), \cos(\alpha))$$

$$\cos(\alpha) = \frac{|a|}{|e'_0|} = \frac{|a|}{1} = |a|$$

Rotation

- Rotates around the z axis by the angle α

2D

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

α – Positive angle to rotate by

Rotation

- Rotates around the z axis by the angle α

2D

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

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3D

- Similar matrices that rotate around each main axis.

Rotation

- Rotates around the z axis by the angle α

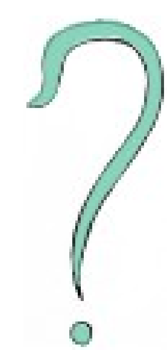
2D

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

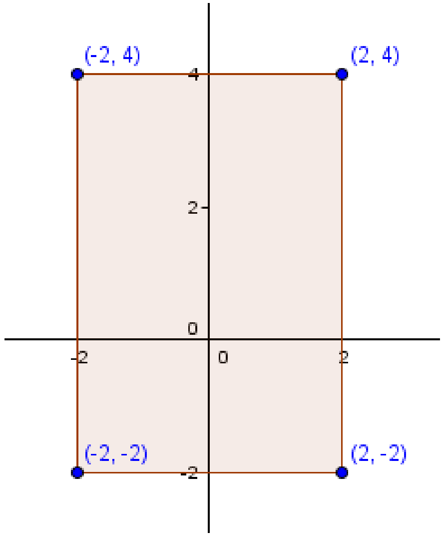
α – Positive angle to rotate by

3D

- Similar matrices that rotate around each main axis.
- What about rotation around an arbitrary axis?

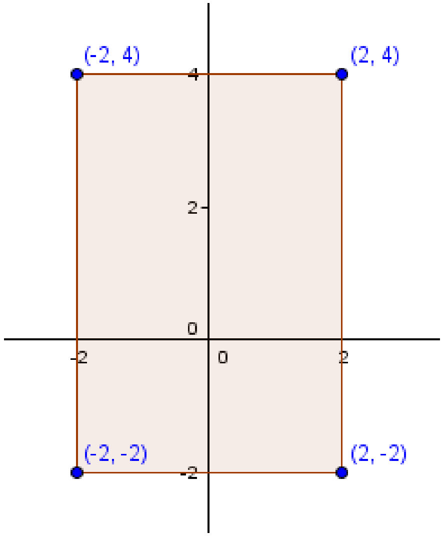


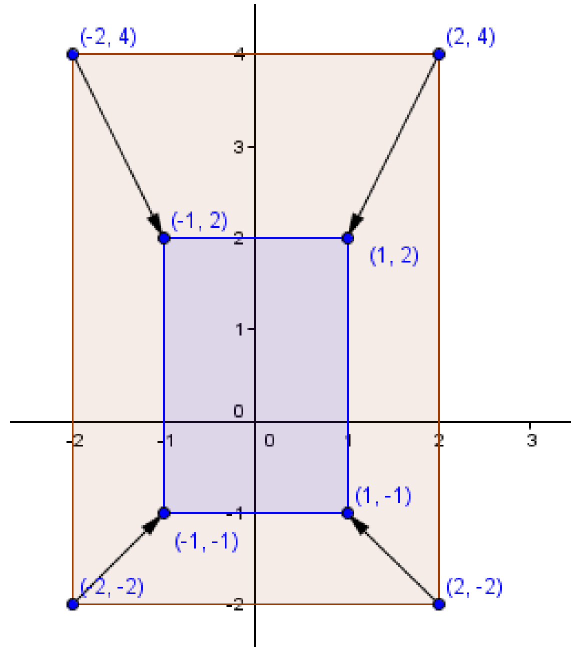
Linear Transformations

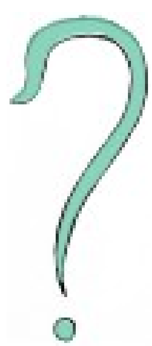


Defined geometry

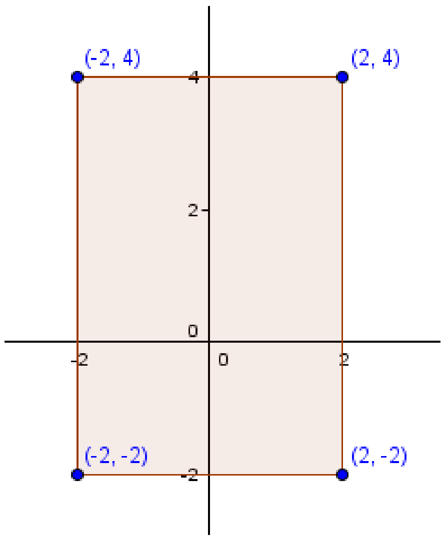
Linear Transformations

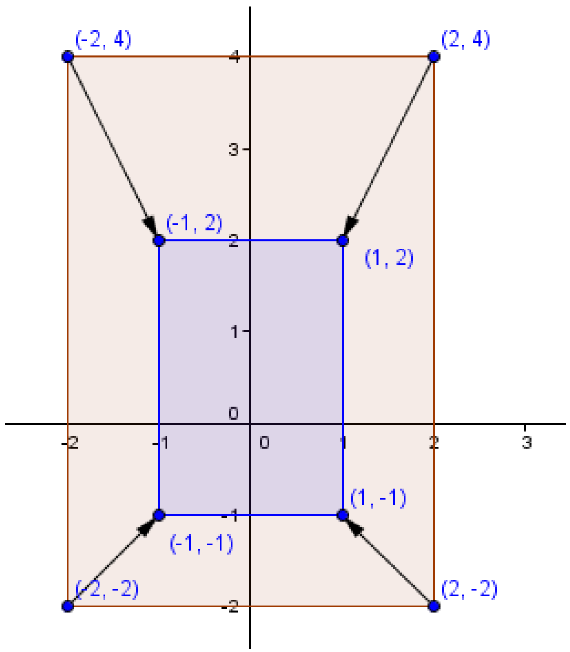






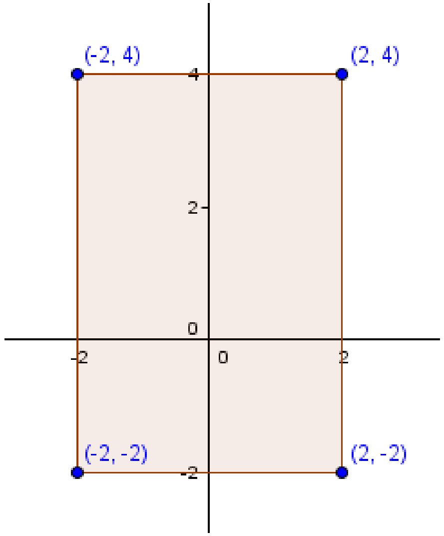
Linear Transformations

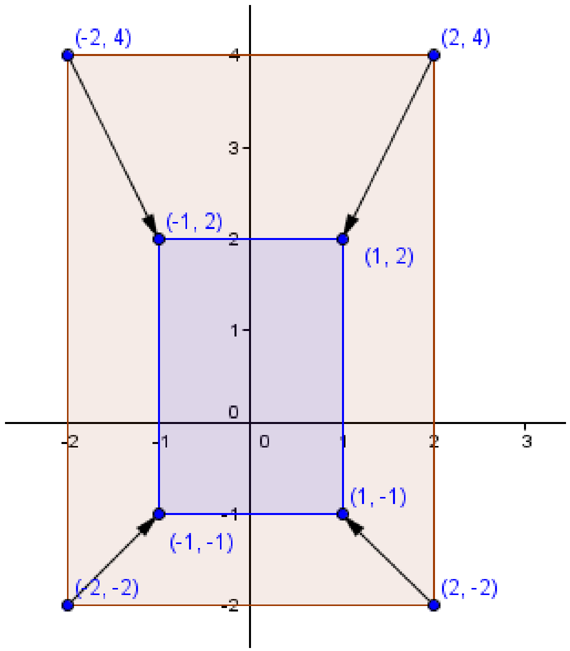


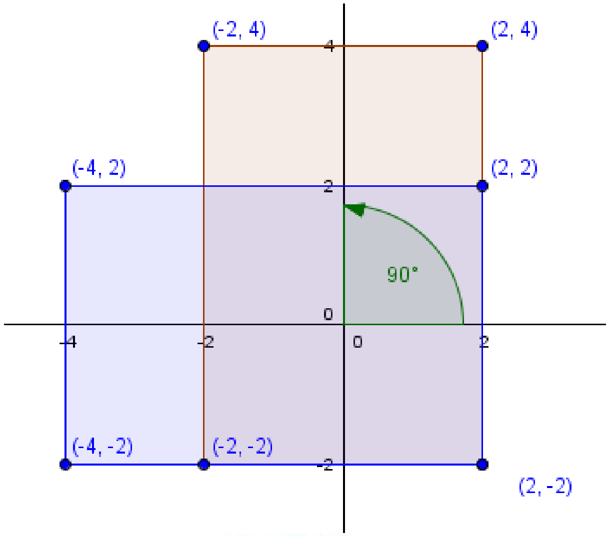


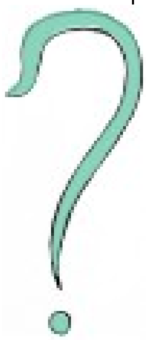
Scale

Linear Transformations

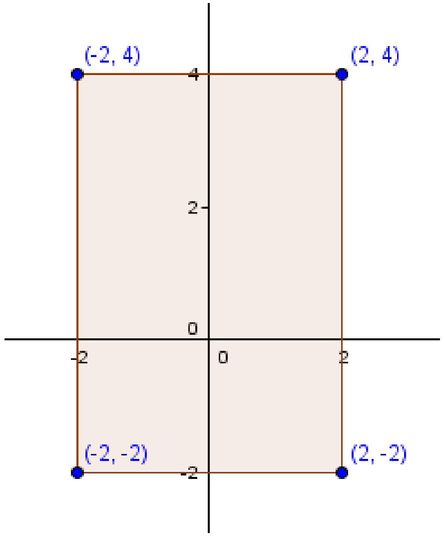


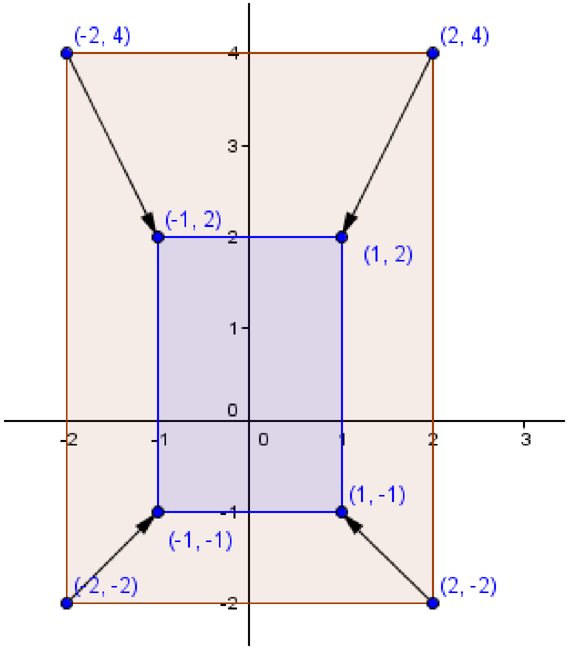


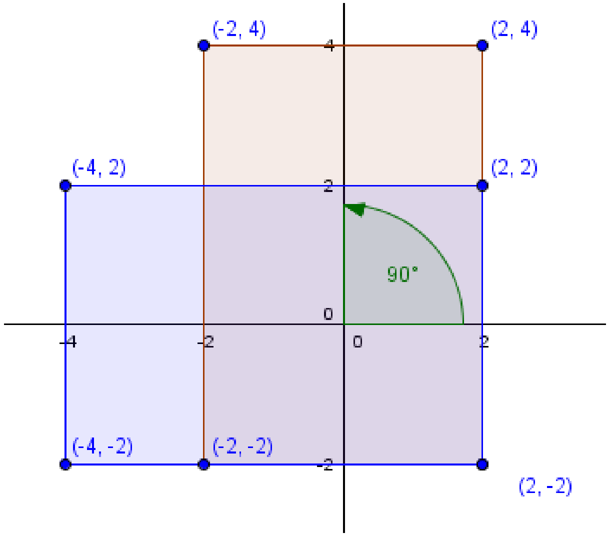




Linear Transformations

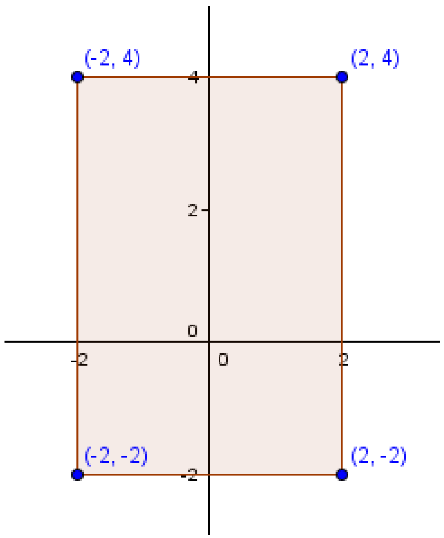


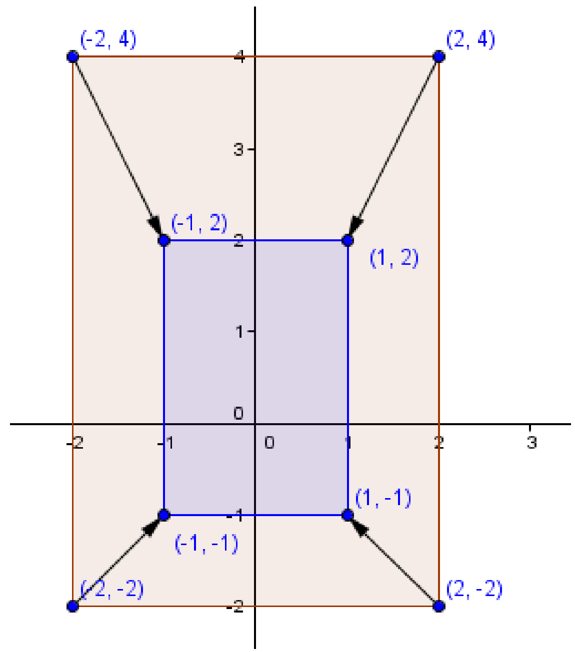


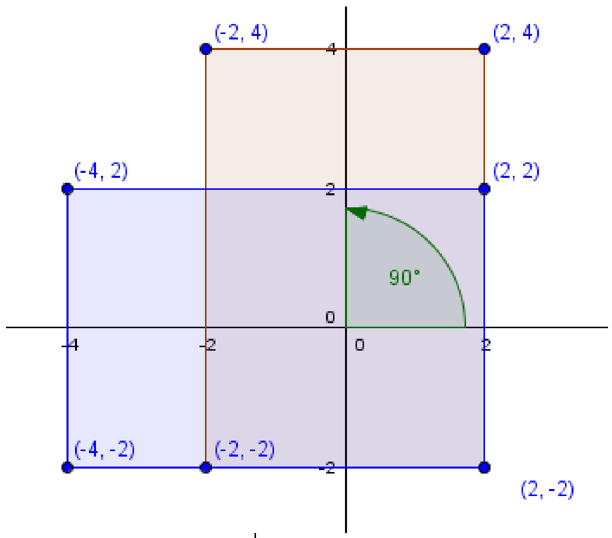


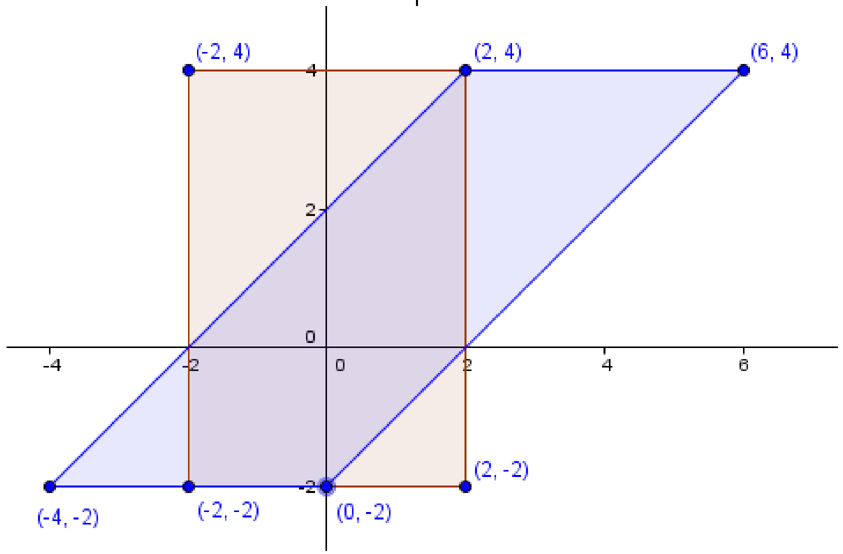
Rotation

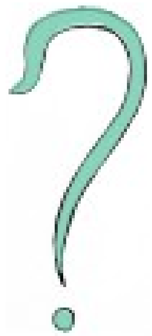
Linear Transformations



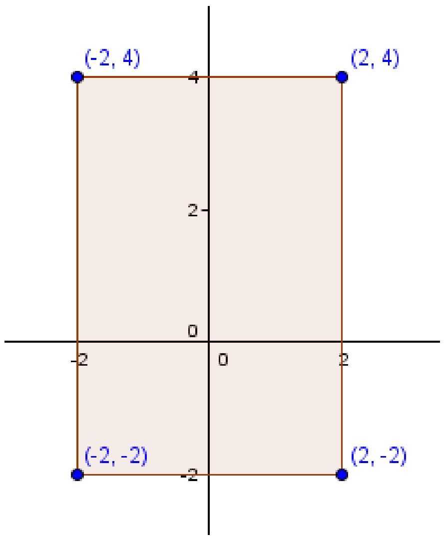


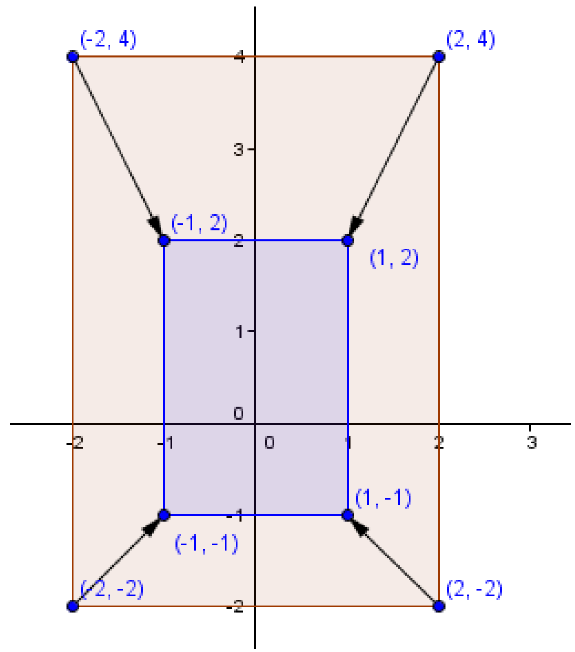


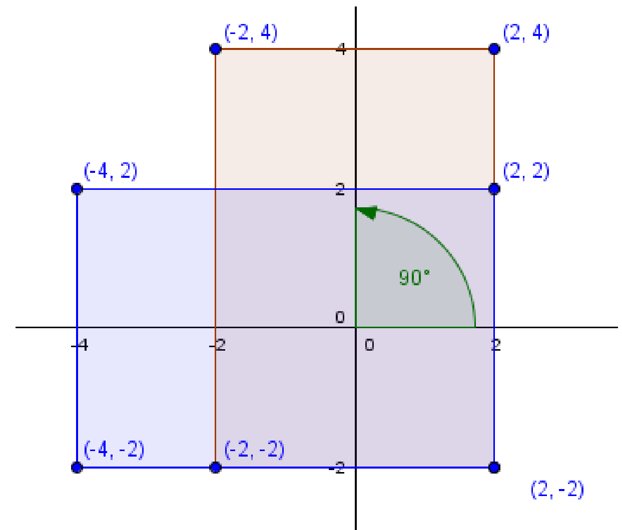


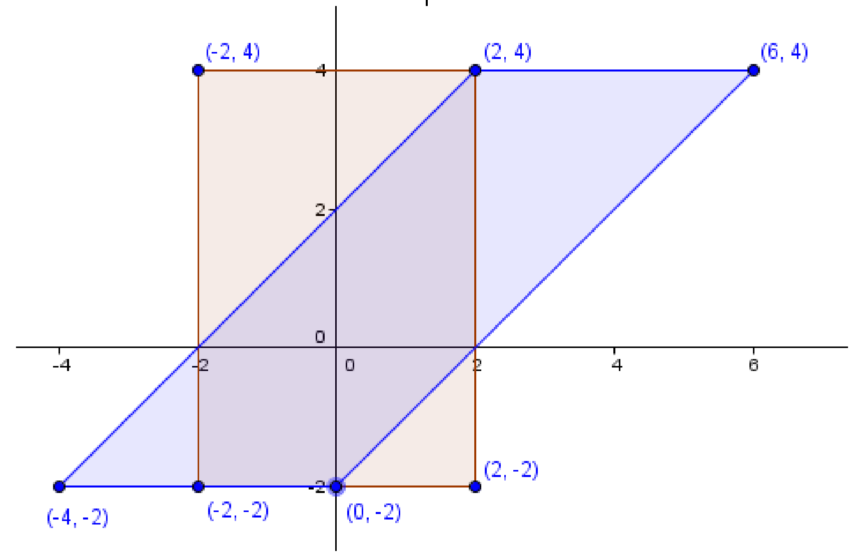


Linear Transformations



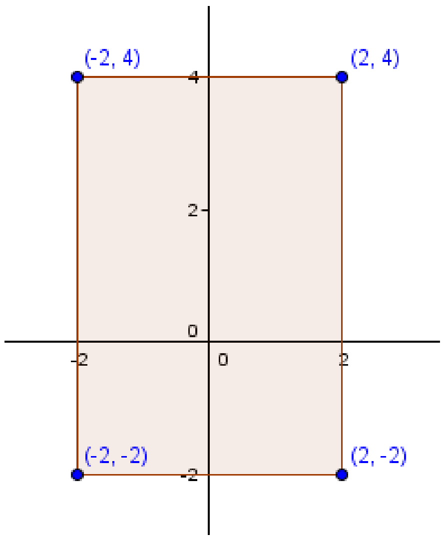


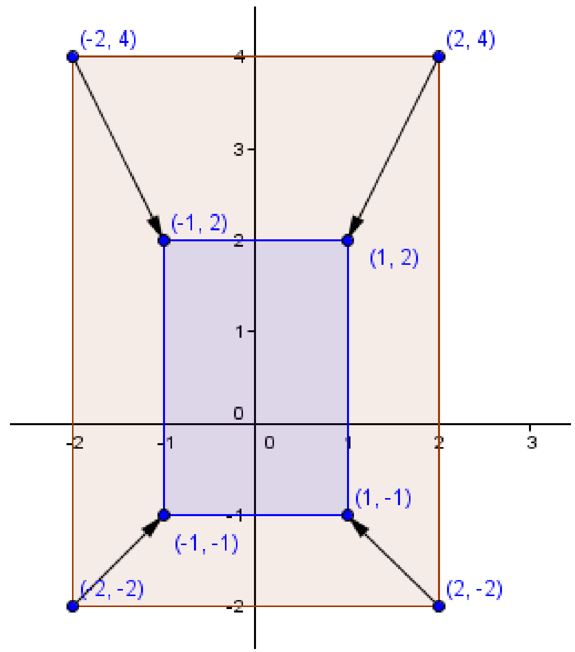


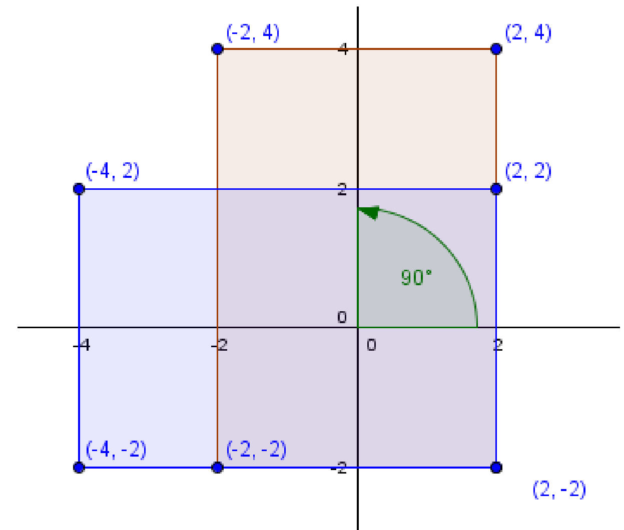


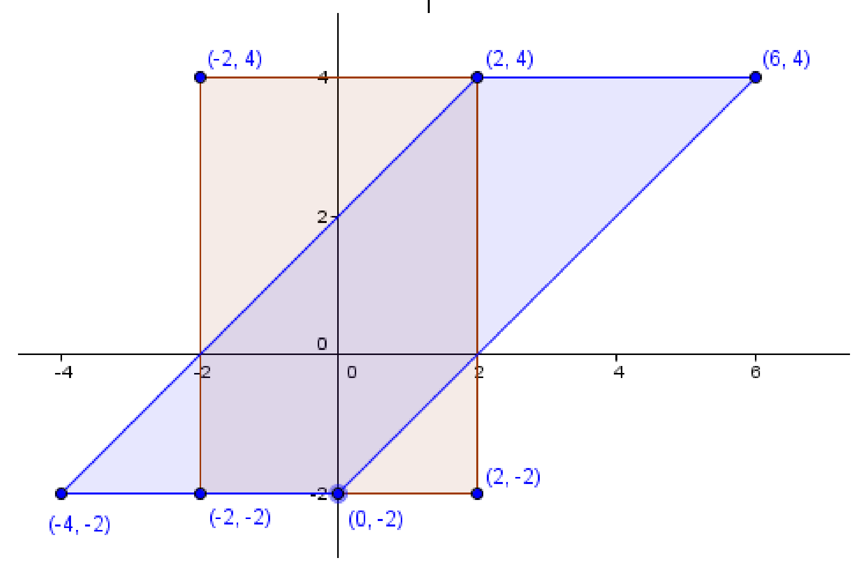
Shear

Linear Transformations









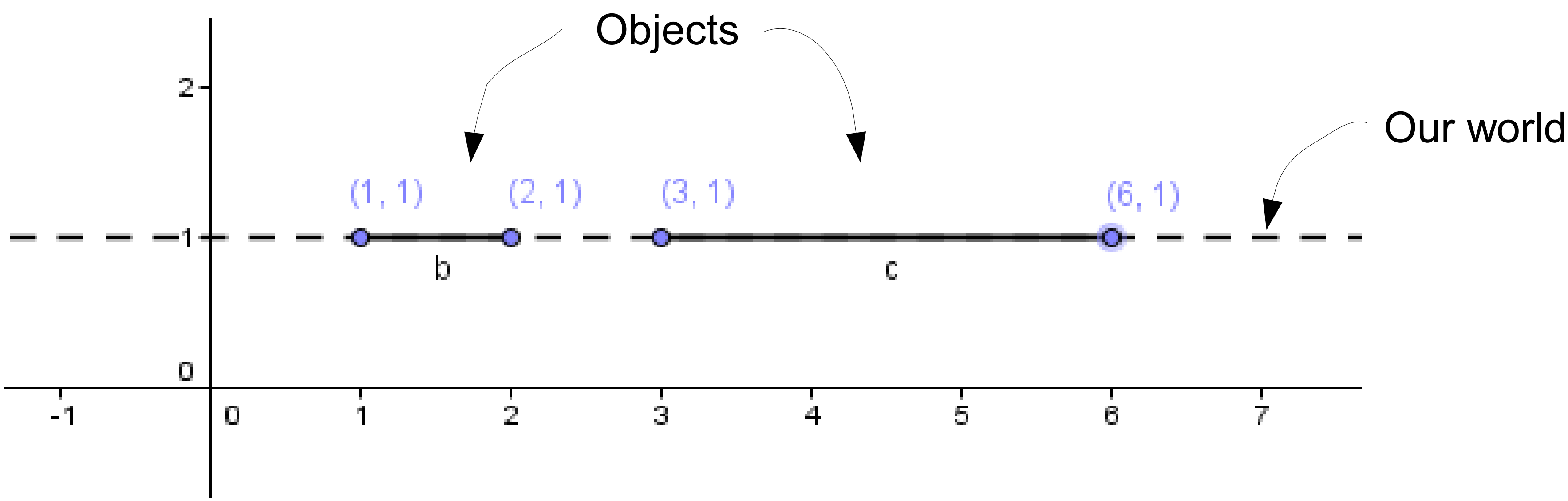
- Will these be enough?

Translation

- Imagine a 1D world located at y=1 line in 2D.

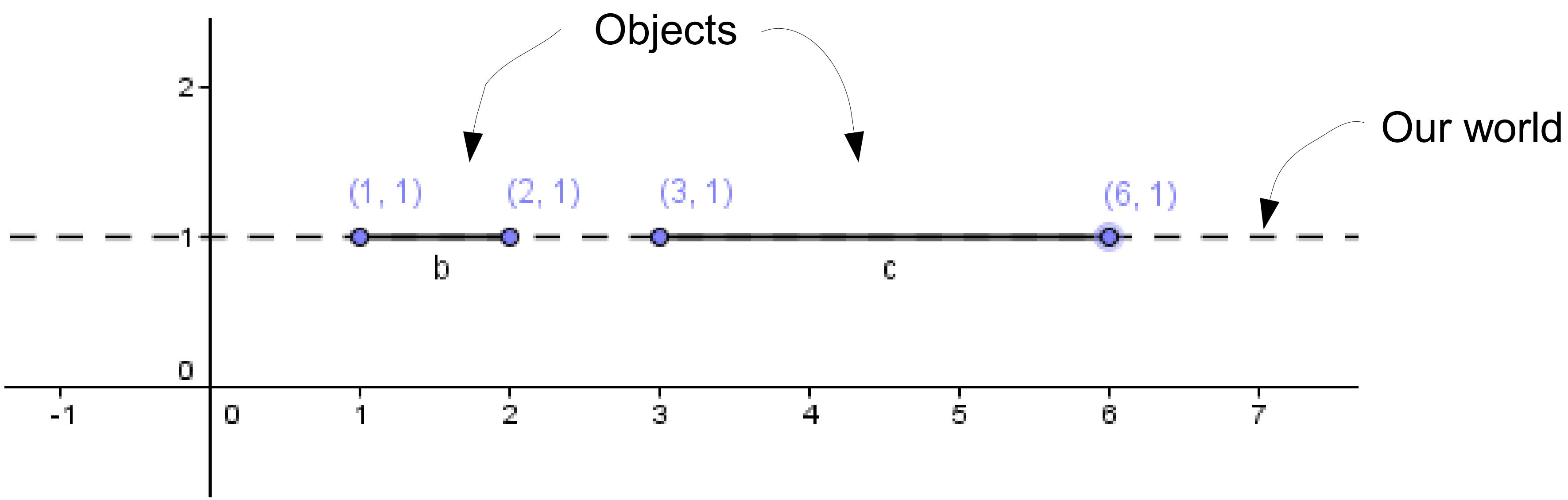
Translation

- Imagine a 1D world located at *y*=1 line in 2D.



Translation

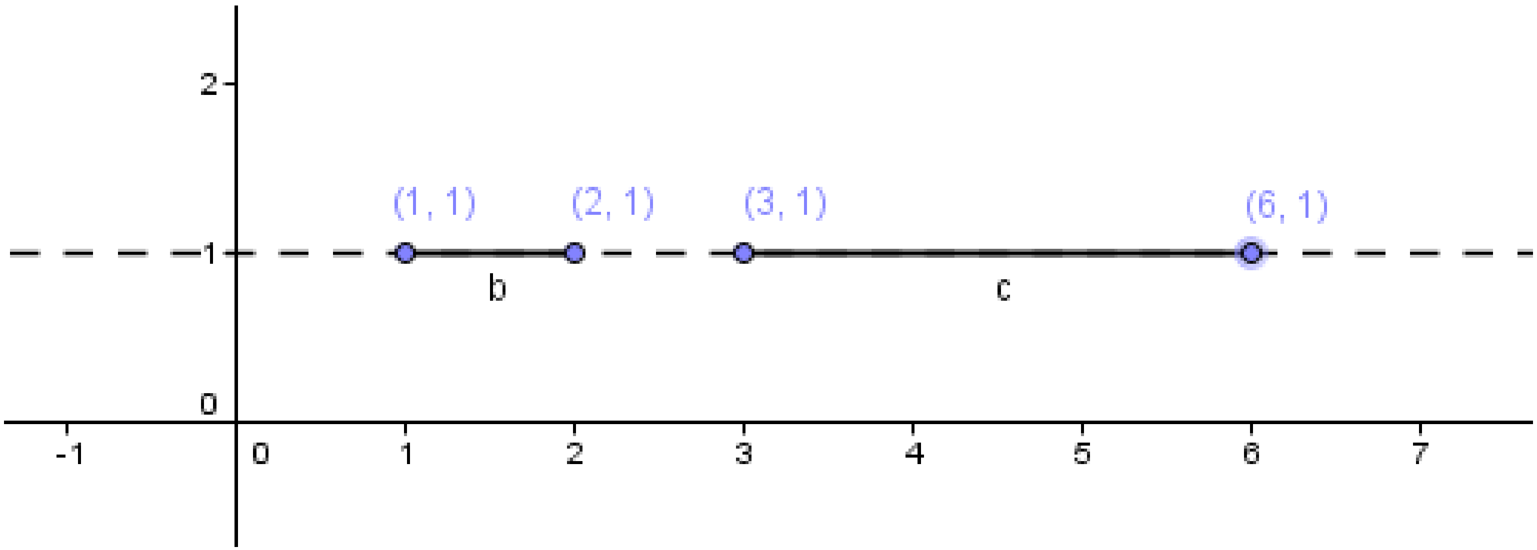
- Imagine a 1D world located at *y*=1 line in 2D.



- Notice that all the points are in the form: (*x*, 1)

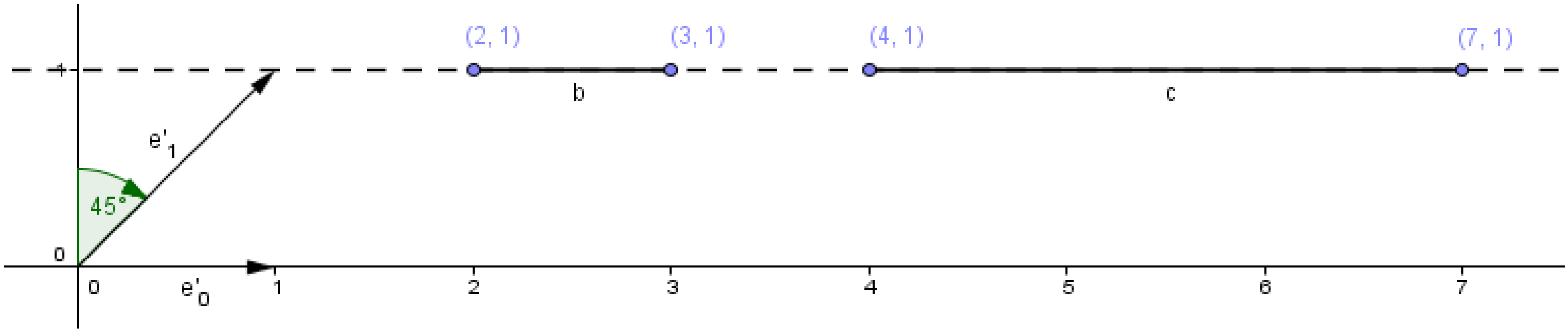
Translation

- How to transform the 2D space so that stuff in the 1D hyperplane y=1 moves an equal amount?

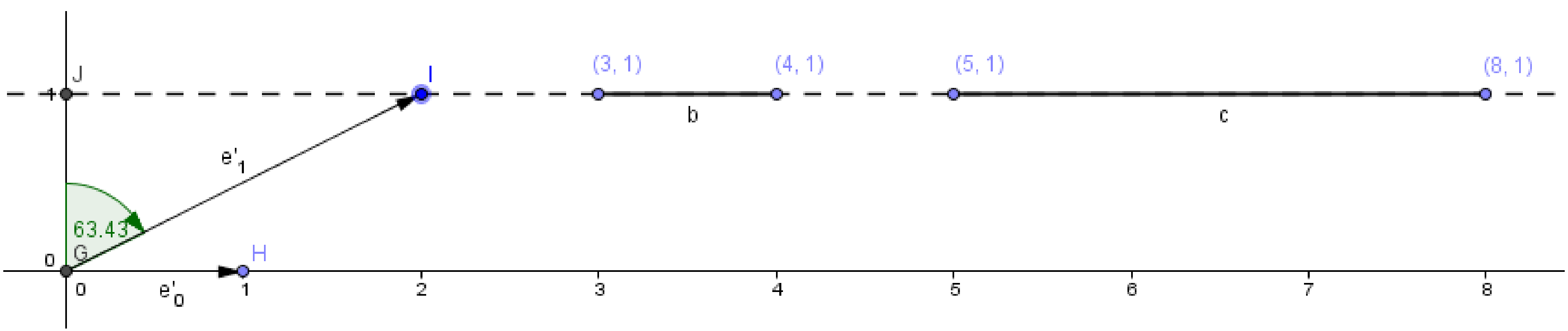


Translation

- Shear-x by tan(45°) = 1



- Shear-x with tan(63.4°) = 2



Translation

- Affine transformation in the current space, linear shear transformation in 1 dimension higher space.

2D

Shear-xy

$$\begin{pmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + x_t \\ y + y_t \\ 1 \end{pmatrix}$$

3D Shear-xyz

1D

Shear-x

$$\begin{pmatrix} 1 & x_t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} x + x_t \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & x_t \\ 0 & 1 & 0 & y_t \\ 0 & 0 & 1 & z_t \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + x_t \\ y + y_t \\ z + z_t \\ 1 \end{pmatrix}$$

Transformations

- This together gives us a **very good toolset** to transform our geometry as we wish.

Affine trasnformation

$$\begin{pmatrix} a & b & c & x_t \\ d & e & f & y_t \\ g & h & i & z_t \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} ax+by+cz+x_t \\ dx+ey+fz+y_t \\ gx+hy+iz+z_t \\ 1 \end{pmatrix}$$

Augmented matrix

Transformations

- This together gives us a **very good toolset** to transform our geometry as we wish.

Linear transformations

Affine trasnformation

$$\begin{pmatrix} a & b & c & x_t \\ d & e & f & y_t \\ g & h & i & z_t \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} ax+by+cz+x_t \\ dx+ey+fz+y_t \\ gx+hy+iz+z_t \\ 1 \end{pmatrix}$$

Augmented matrix

Transformations

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Linear transformations

Translation column

Affine trasnformation

$$\begin{pmatrix} a & b & c & x_t \\ d & e & f & y_t \\ g & h & i & z_t \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} ax + by + cz + x_t \\ dx + ey + fz + y_t \\ gx + hy + iz + z_t \\ 1 \end{pmatrix}$$

Augmented matrix

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Linear transformations

Translation column

Affine trasnformation

$$\begin{pmatrix} a & b & c & x_t \\ d & e & f & y_t \\ g & h & i & z_t \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} ax + by + cz + x_t \\ dx + ey + fz + y_t \\ gx + hy + iz + z_t \\ 1 \end{pmatrix}$$

Used for perspective projection...

Augmented matrix

Multiple Transformations

- How can we apply multiple transformations?

$$A \cdot \left(B \cdot \left(C \cdot v \right) \right)$$

- Is it the same as?

$$B \cdot \left(A \cdot \left(C \cdot v \right) \right)$$

Transformations

- In some graphics libraries you assign the **position / translation**, **rotation** and **scale** individually.

Transformations

- In some graphics libraries you assign the **position / translation**, **rotation** and **scale** individually.

```
object.position.set(2.7, 1.2, 0);
object.scale.set(2.4, 0.1, 0.4);
object.rotation.set(0, toRad(180), 0);
```

Transformations

- In some graphics libraries you assign the **position / translation**, **rotation** and **scale** individually.
- To the GPU the object transformations are sent as a matrix (*model matrix*).

Transformations

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- To the GPU the object transformations are sent as a matrix (*model matrix*).

$$\mathrm{projectionMatrix} \cdot \mathrm{viewMatrix} \cdot \mathrm{modelMatrix} \cdot v$$

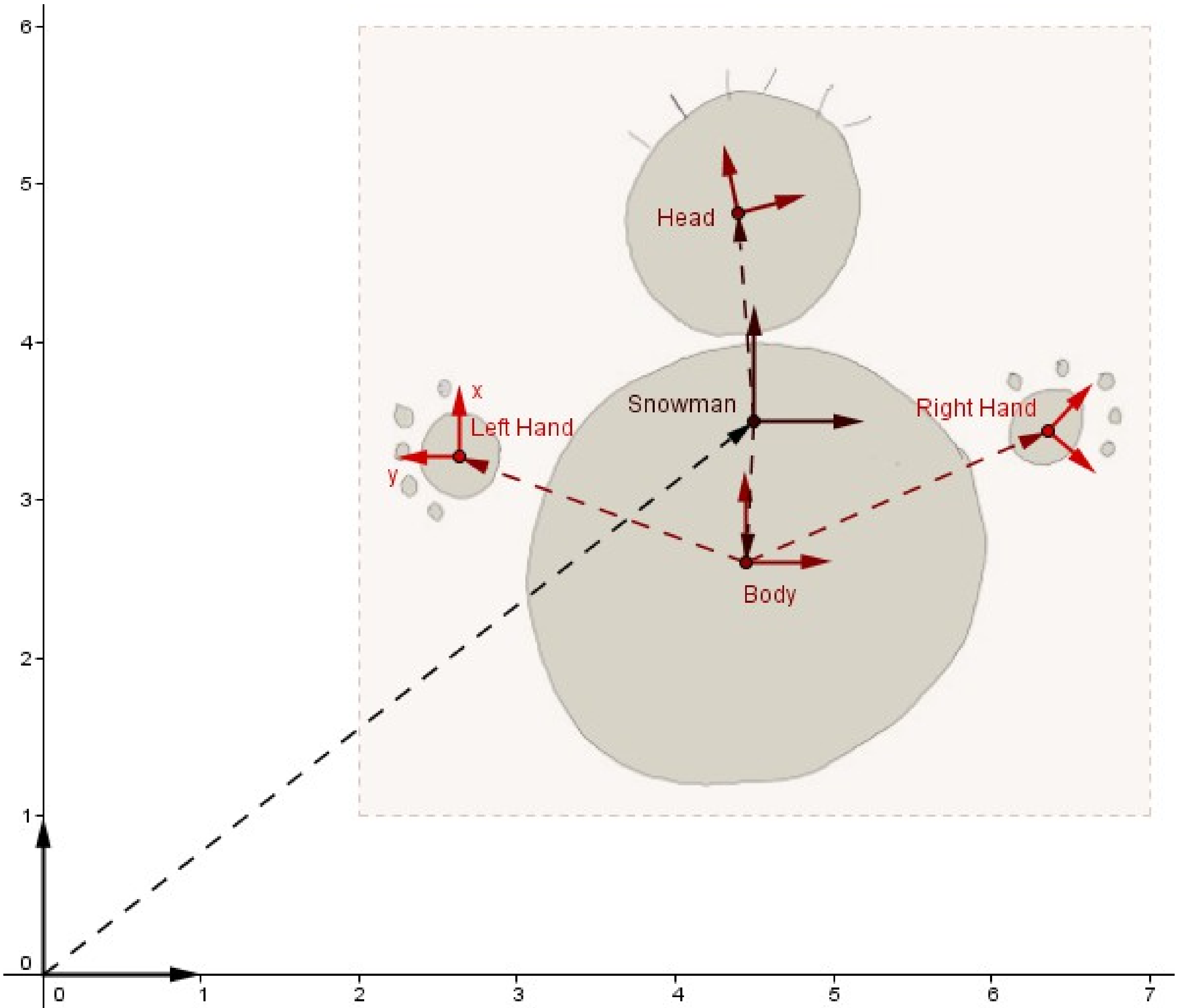
$$P \cdot V \cdot M \cdot v$$

Transformations

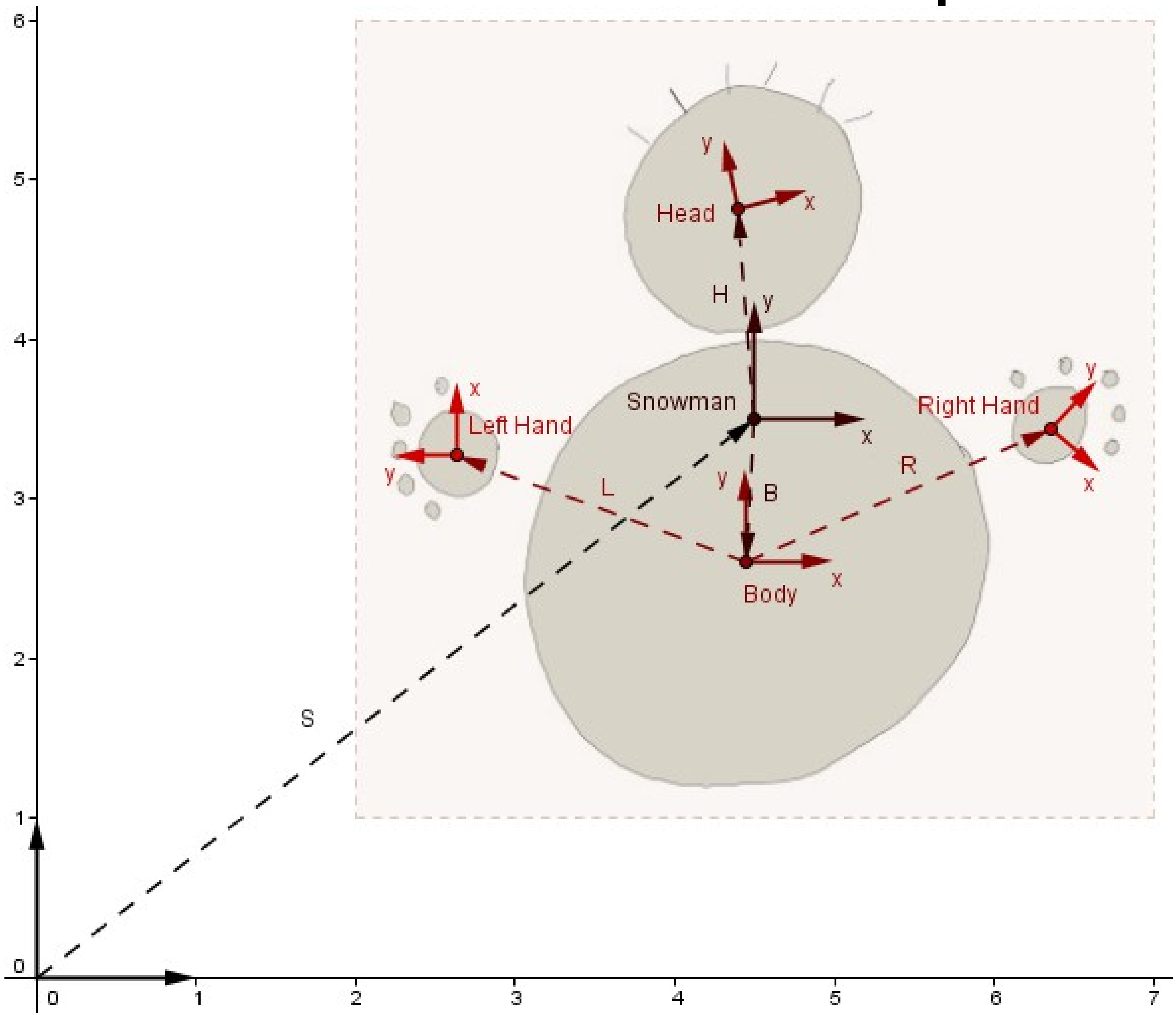
- In some graphics libraries you assign the **position / translation**, **rotation** and **scale** individually.
- To the GPU the object transformations are sent as a matrix (*model matrix*).
- Questions about transformations?

Scene Graph

- Dependency between (parts of) objects.



Scene Graph



Head

$S \cdot H \cdot v$

Body

$S \cdot B \cdot v$

Left hand

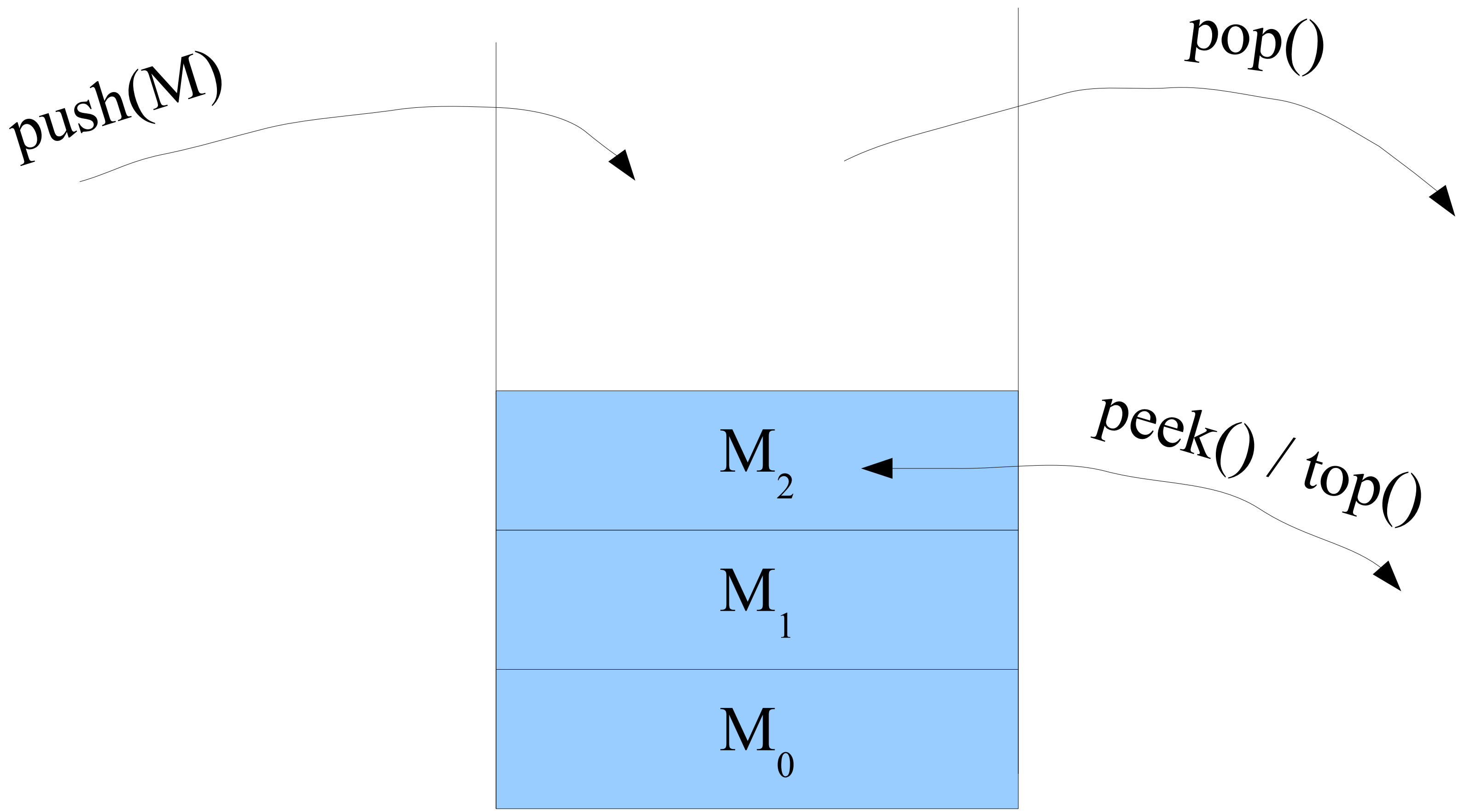
$S \cdot B \cdot L \cdot v$

Right hand

$S \cdot B \cdot R \cdot v$

Matrix Stack

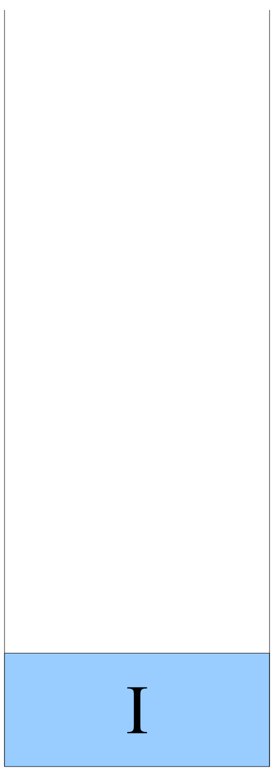
- Stack can be used to save and load matrices (intermediary states)



Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
- *Current state* is in the **top of the stack**

1) M = Identity, push(M)



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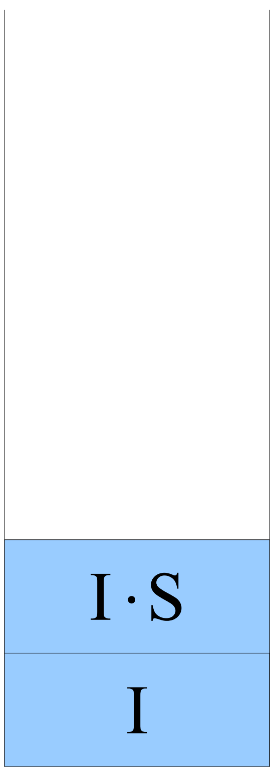
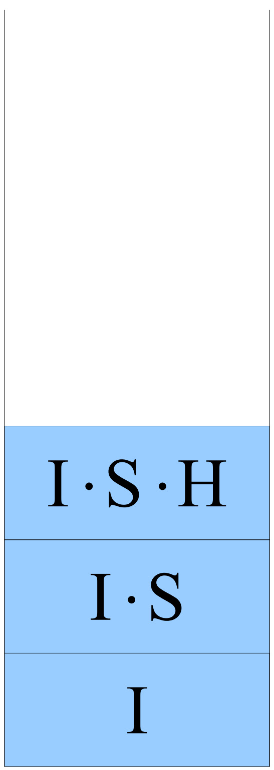


1) M = Identity, push(M)

2) M *= S, push(M) Move to snowman's space

Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
- *Current state* is in the **top of the stack**



1) M = Identity, push(M)
2) M *= S, push(M)
3) M *= H, push(M)

Move to head's space

Matrix Stack

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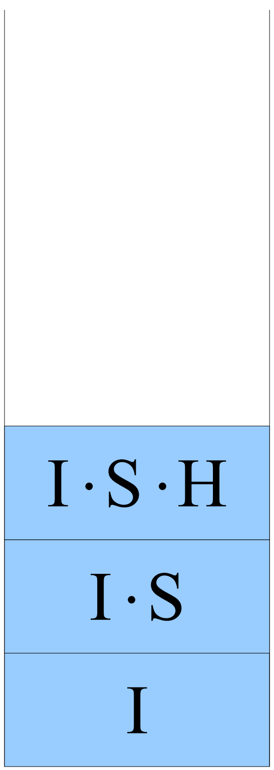

1) M = Identity, push(M)
2) M *= S, push(M)
3) M *= H, push(M)
4) *Draw head vertices*

Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
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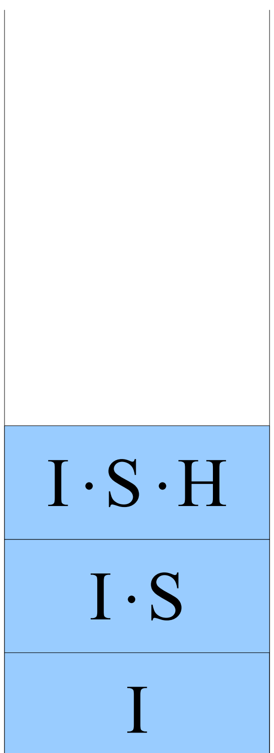
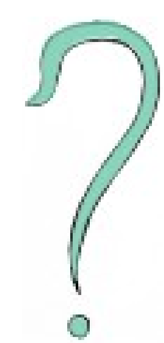


1) M = Identity, push(M)
2) M *= S, push(M)
3) M *= H, push(M)
4) *Draw head vertices*

We now want to get back to the snowman's space



Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
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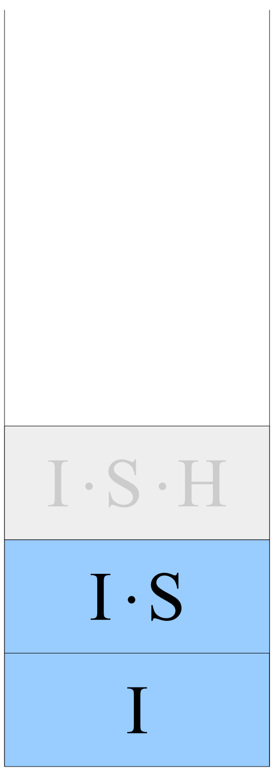

1) M = Identity, push(M)
2) M *= S, push(M)
3) M *= H, push(M)
4) *Draw head vertices*
5) pop(), M = top()

Matrix Stack

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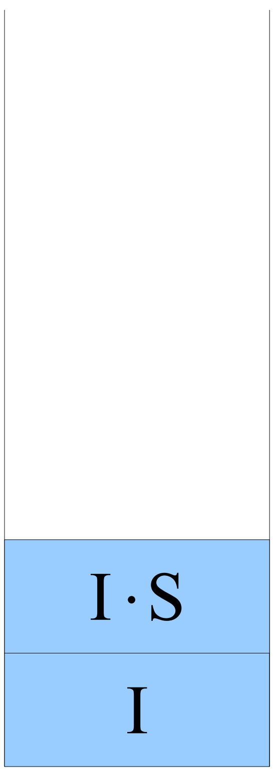
- *Current state* is in the **top of the stack**



2) ...

3) M *= H, push(M)

4) *Draw head vertices*

5) pop(), M = top()

Matrix Stack

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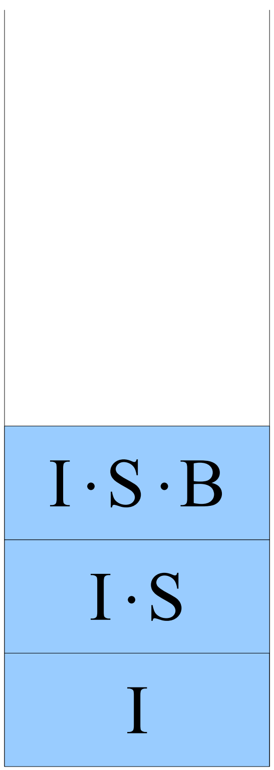
- *Current state* is in the **top of the stack**



2) ...

3) M *= H, push(M)

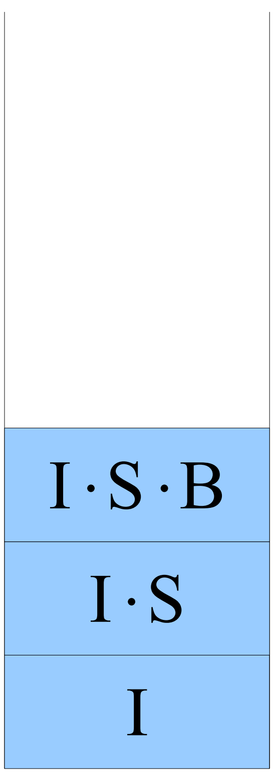
4) *Draw head vertices*

5) pop(), M = top()

6) M *= B, push(M) Move to body's space

Matrix Stack

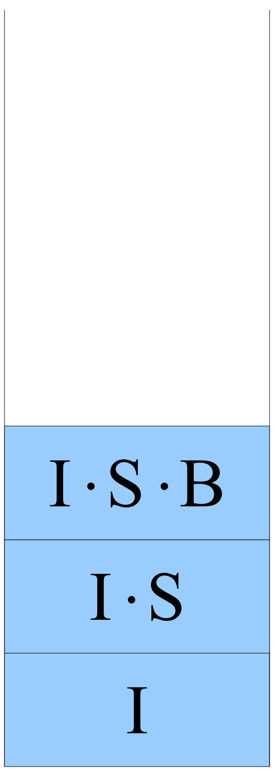
- Stack can be used to save and load matrices (intermediary states)
- *Current state* is in the **top of the stack**



2) ...
3) M *= H, push(M)
4) *Draw head vertices*
5) pop(), M = top()
6) M *= B, push(M)
7) *Draw body vertices*

Matrix Stack

- Stack can be used to save and load matrices (intermediary states)
- *Current state* is in the **top of the stack**

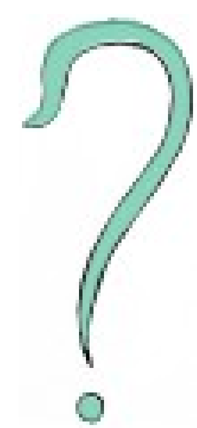


5) ...

6) M *= B, push(M)

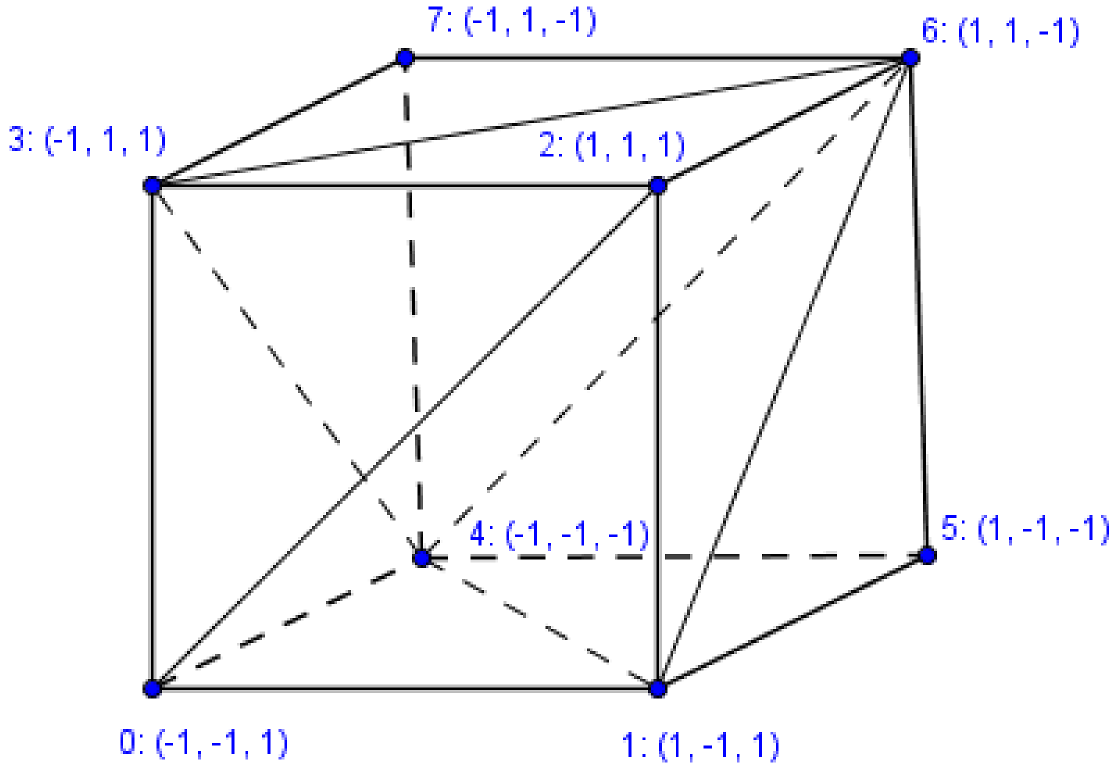
7) *Draw body vertices*

8) *...* ?



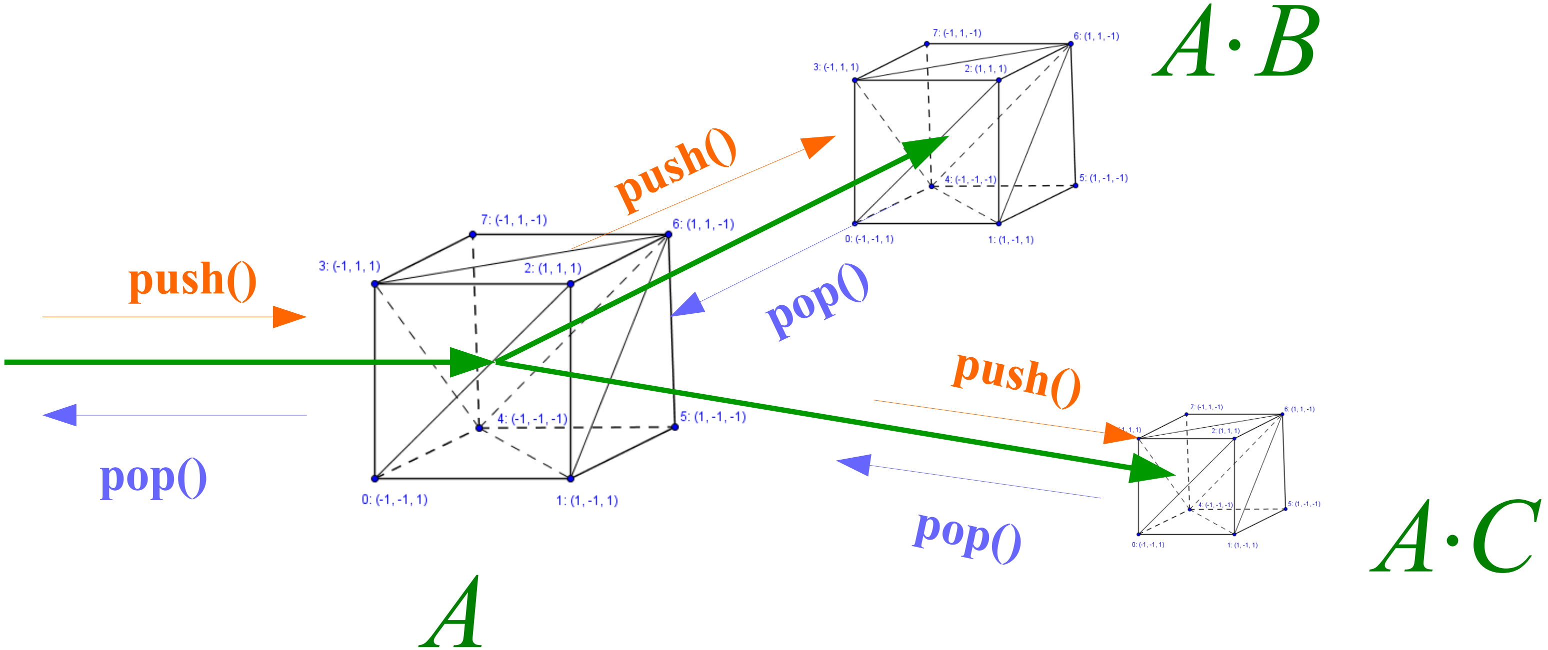
Matrix Stack

- Each (part of an) **object** can be modelled in its own **local space**.



Matrix Stack

- Each (part of an) **object** can be modelled in its own **local space**.
- When we traverse the scene graph, important intermediary states are saved / loaded.



Matrix Stack

- Each (part of an) **object** can be modelled in its own **local space**.
- When we traverse the scene graph, important intermediary states can saved / loaded.
- No need to recalculate same matrix multiplications many times or find inverse transformations.

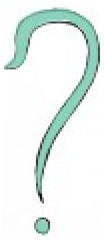
$$M = A \cdot B \cdot D \cdot D^{1} = A \cdot B$$

vs

$$\text{stack.pop()}, \quad M = \text{stack.top}()$$

Matrix Stack

- Each (part of an) **object** can be modelled in its own **local space**.
- When we traverse the scene graph, important intermediary states can saved / loaded.
- No need to recalculate same matrix multiplications many times or find inverse transformations.
- Questions about the matrix stack?



What new did you find out today?

What more would you like to know?

Next time

Frames of reference, projections