

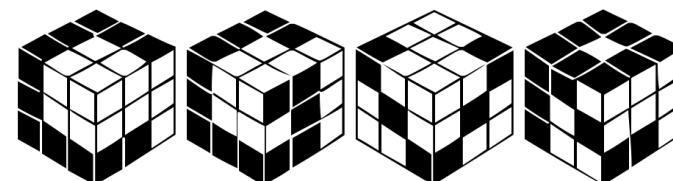
# Computer Graphics

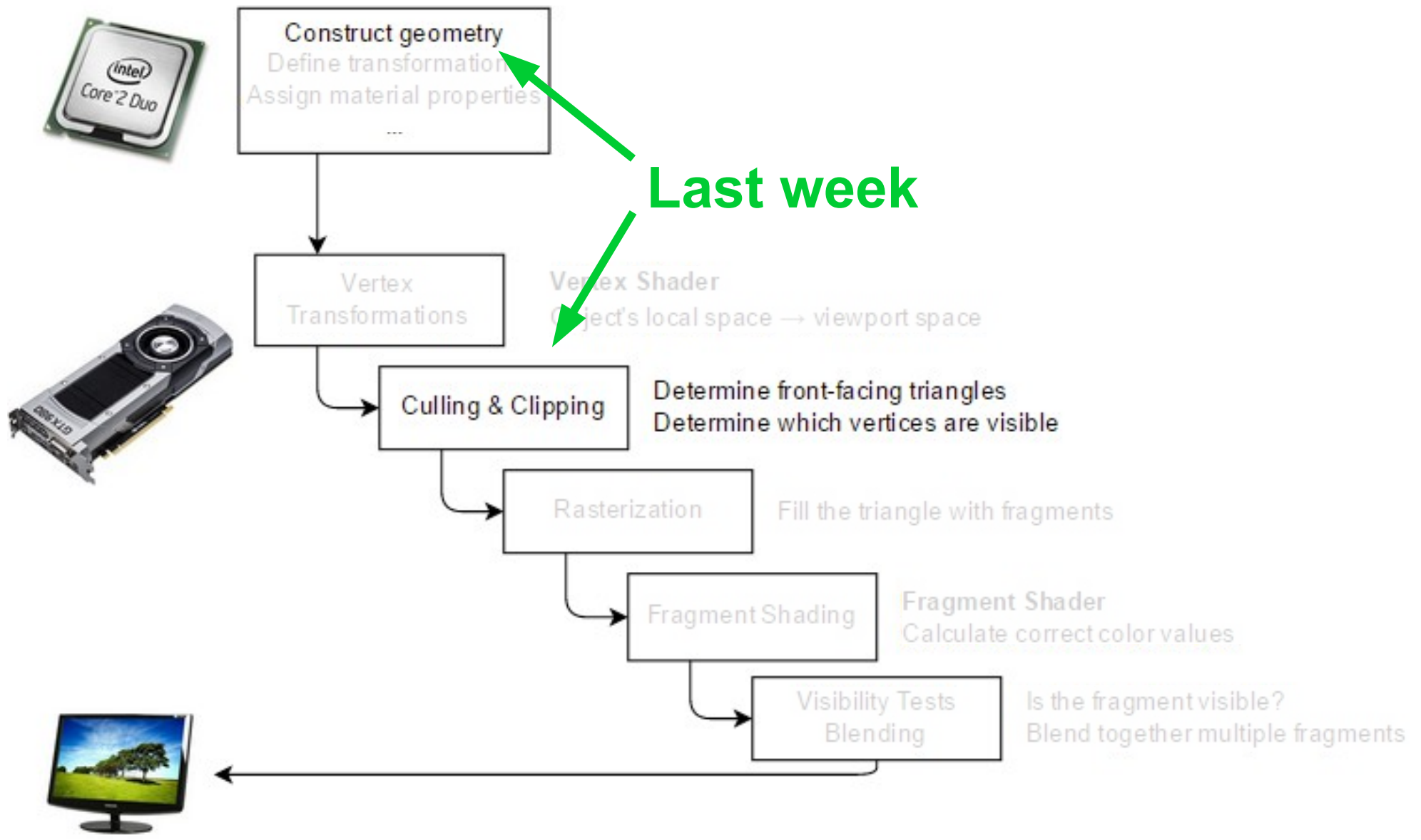
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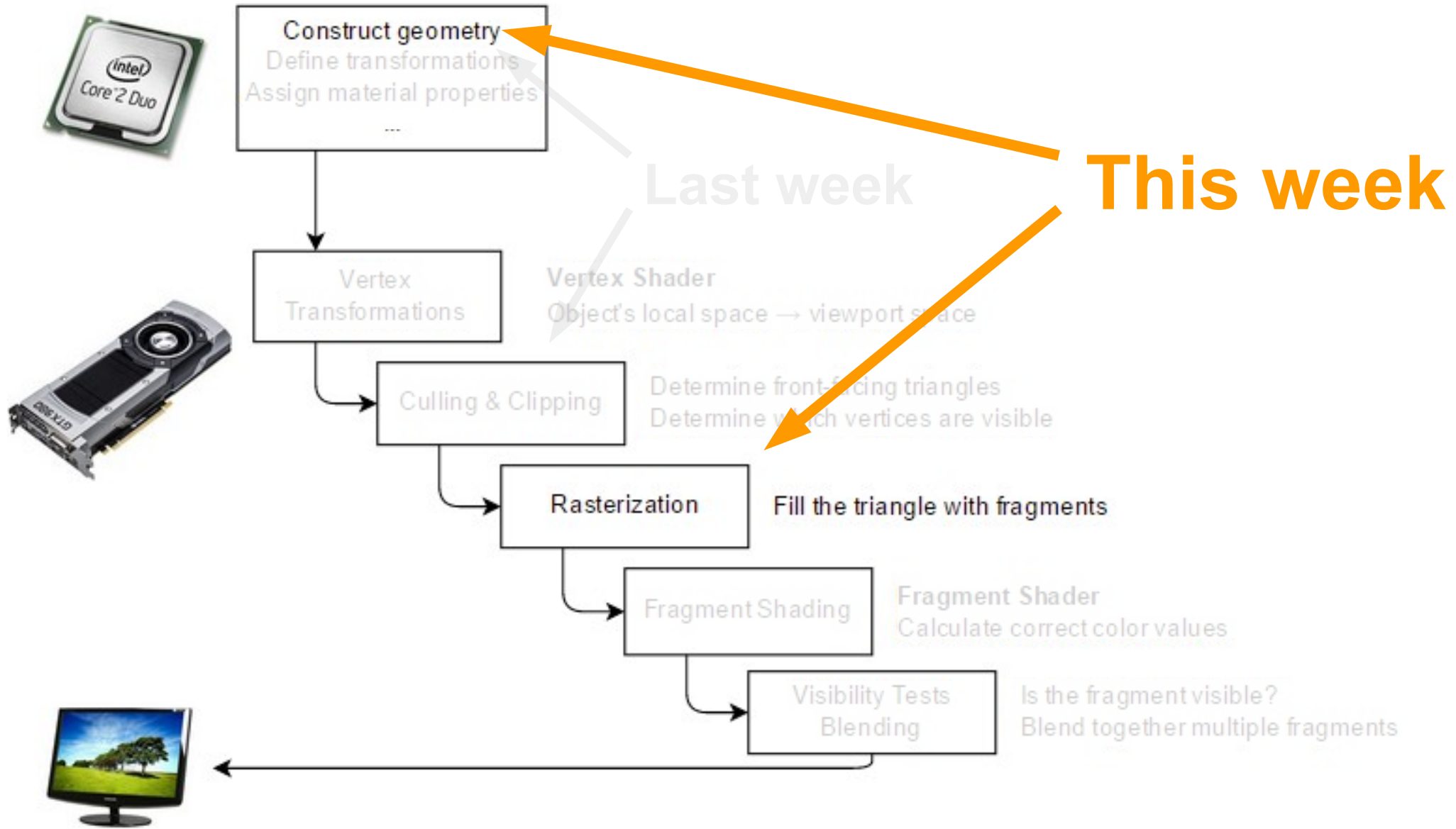
Raimond Tunnel



UNIVERSITY OF TARTU  
Institute of Computer Science

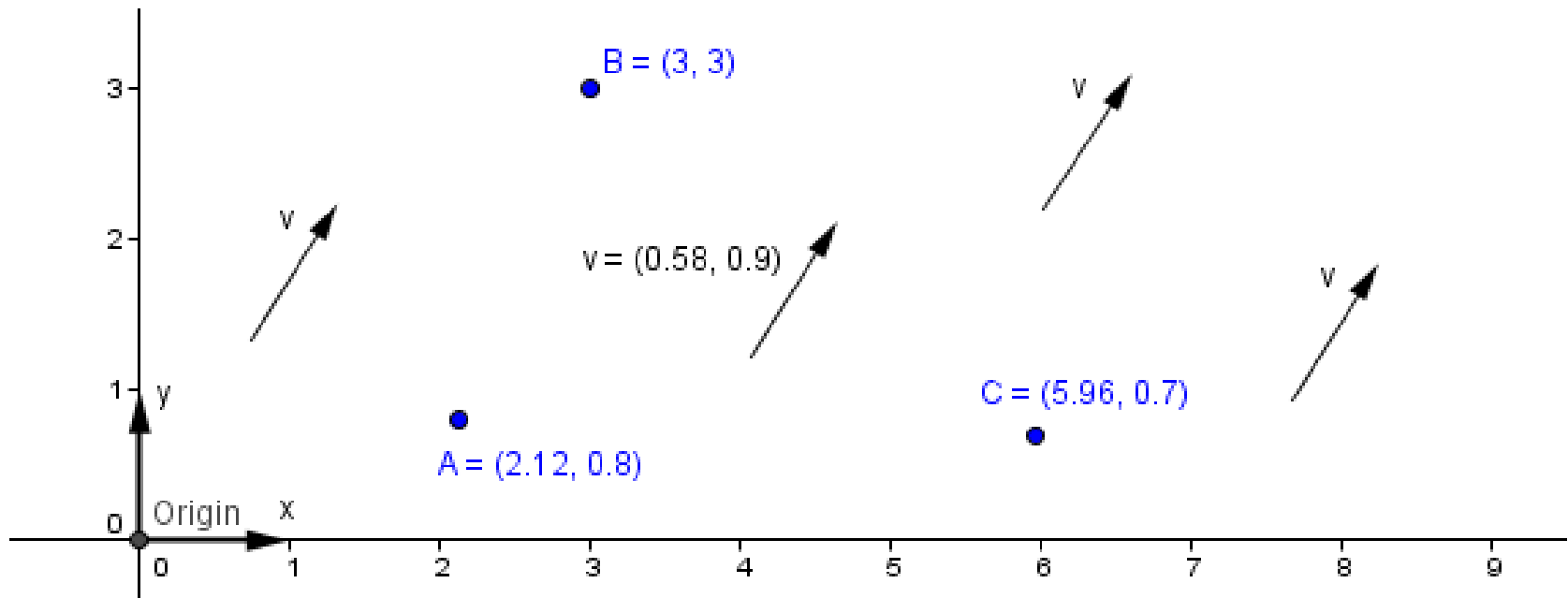






# Points and Vectors

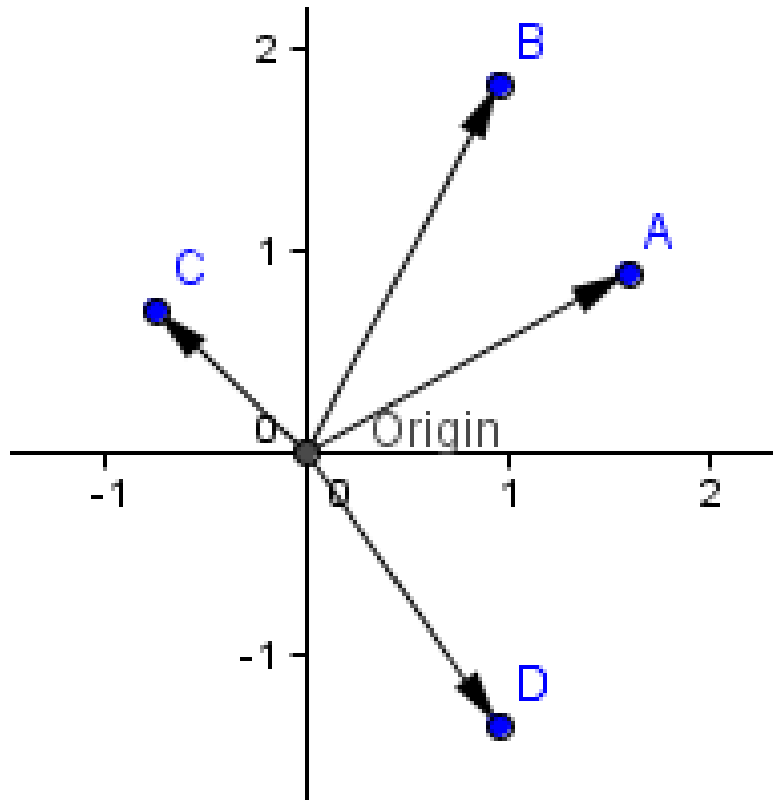
- In computer graphics we distinguish:
  - **Point** – a location in space (location vector, *kohavektor*)
  - **Vector** – a direction in space (direction vector, *suunavektor*)



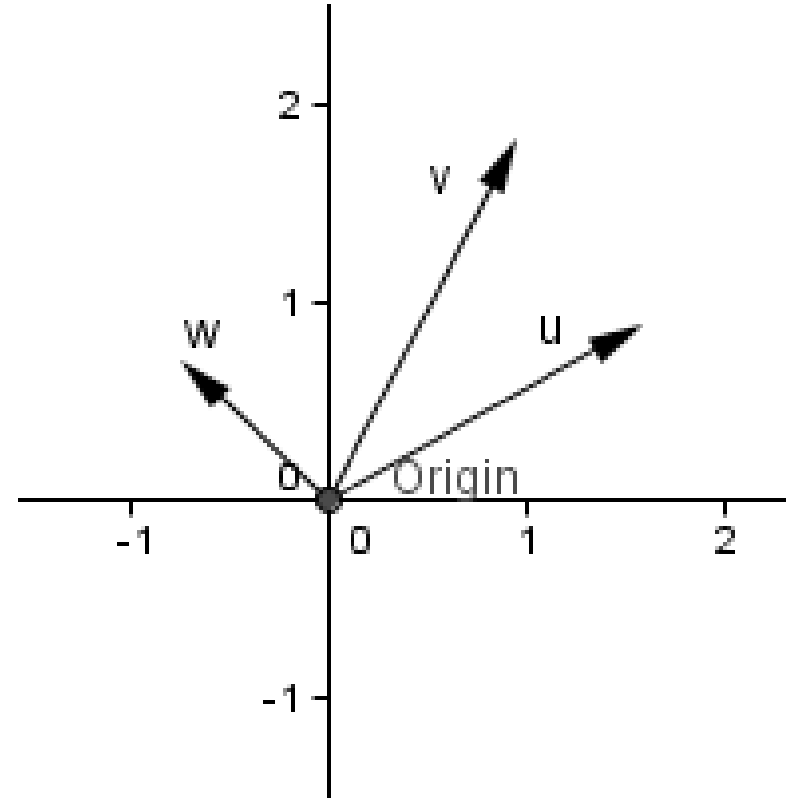
# Points and Vectors

- Both are elements of a 2-, 3- or 4-dimensional **vector space** (*vektorruum*) over the field (*korpus*)  $\mathbb{R}$ .
- More precisely, elements of a **coordinate space**.
- So both are *vectors* in terms of algebra.
- Some operations make sense for vectors, some for points.
- A space that contains them both and defines an addition between a point and a vector is called an **affine space**.
- More precisely, an **Euclidean space**.

# Points and Vectors



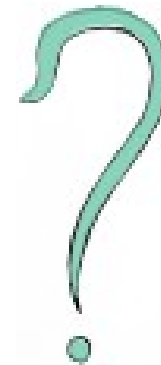
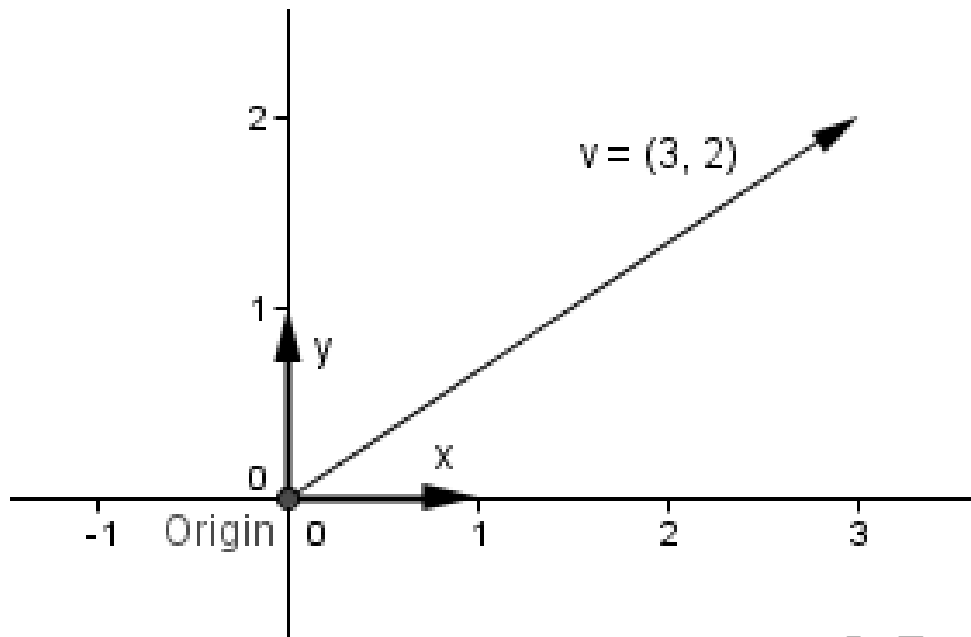
Vector space of points



Vector space of vectors

# Points and Vectors

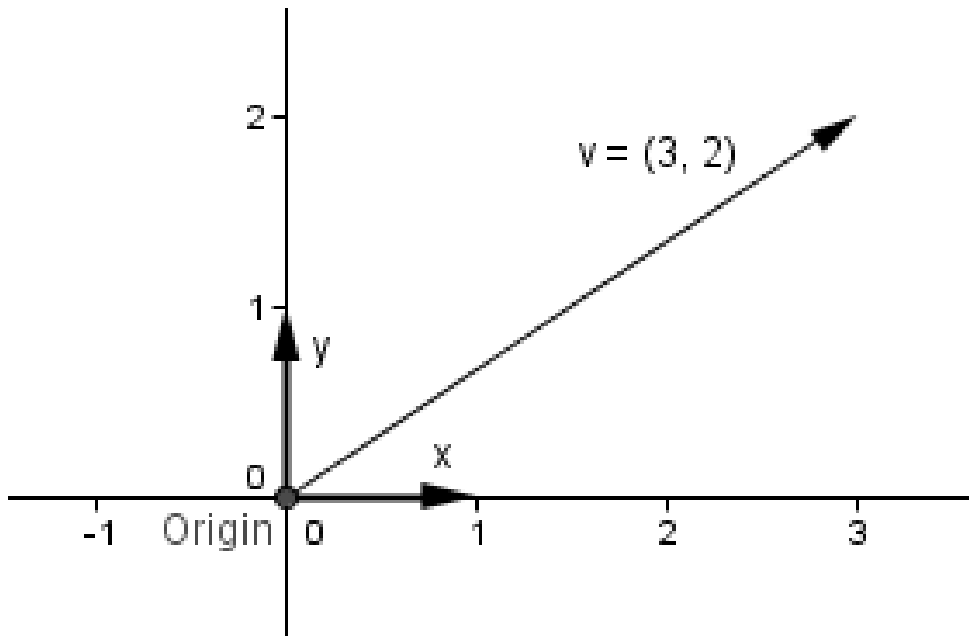
- Given a vector space over  $\mathbb{R}^2$  with a **basis** (*baas*) and the **origin**, all the elements of the vector space can be represented as a ... of the ... vectors.



In Estonian: *baas + koordinaatide alguspunkt = reeper*

# Points and Vectors

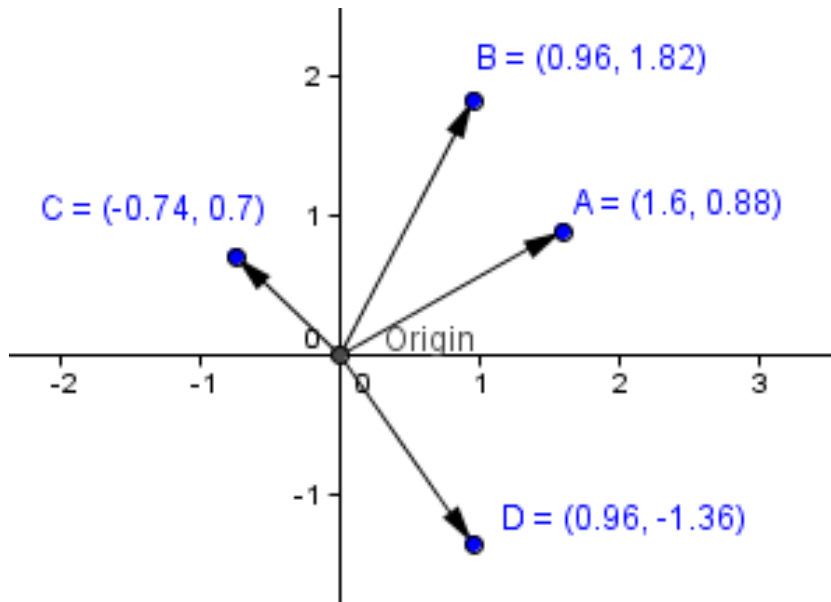
$$v = \alpha_0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



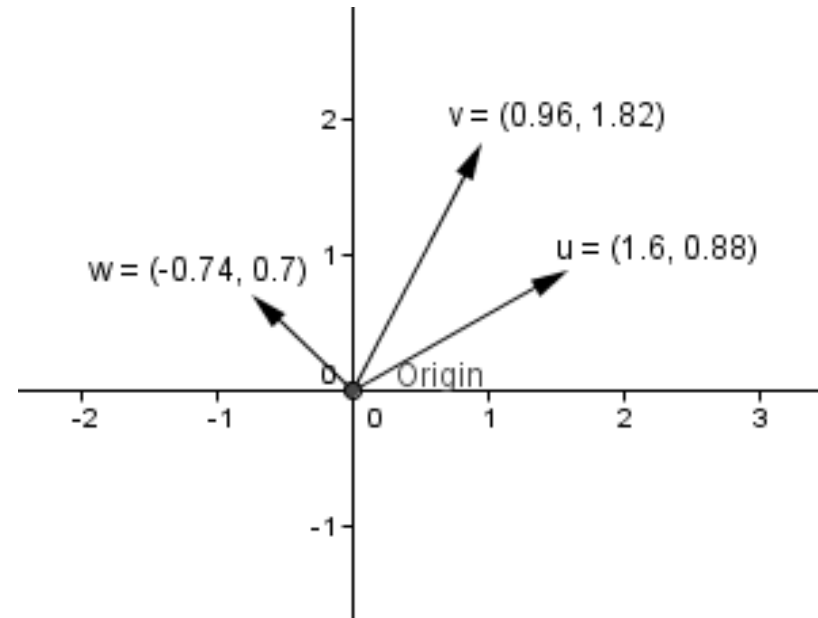


# Points and Vectors

- As the elements of our vector space are  $n$ -tuples, we can call it a **coordinate space**.



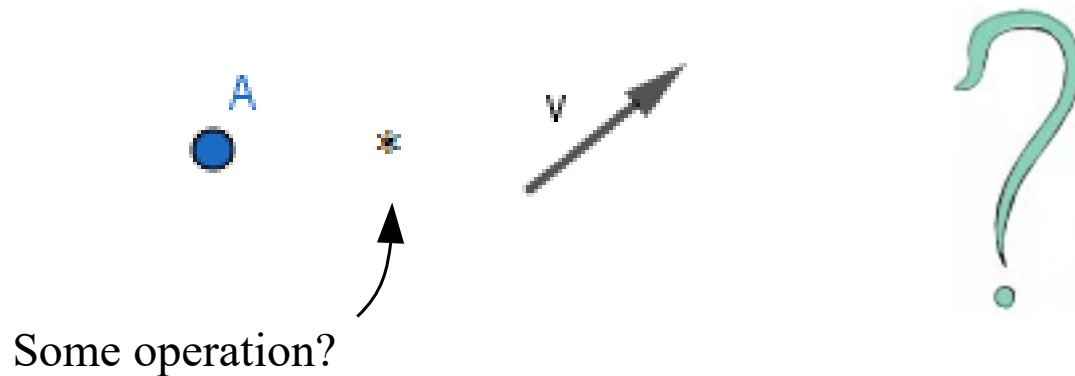
Coordinate space of points



Coordinate space of vectors

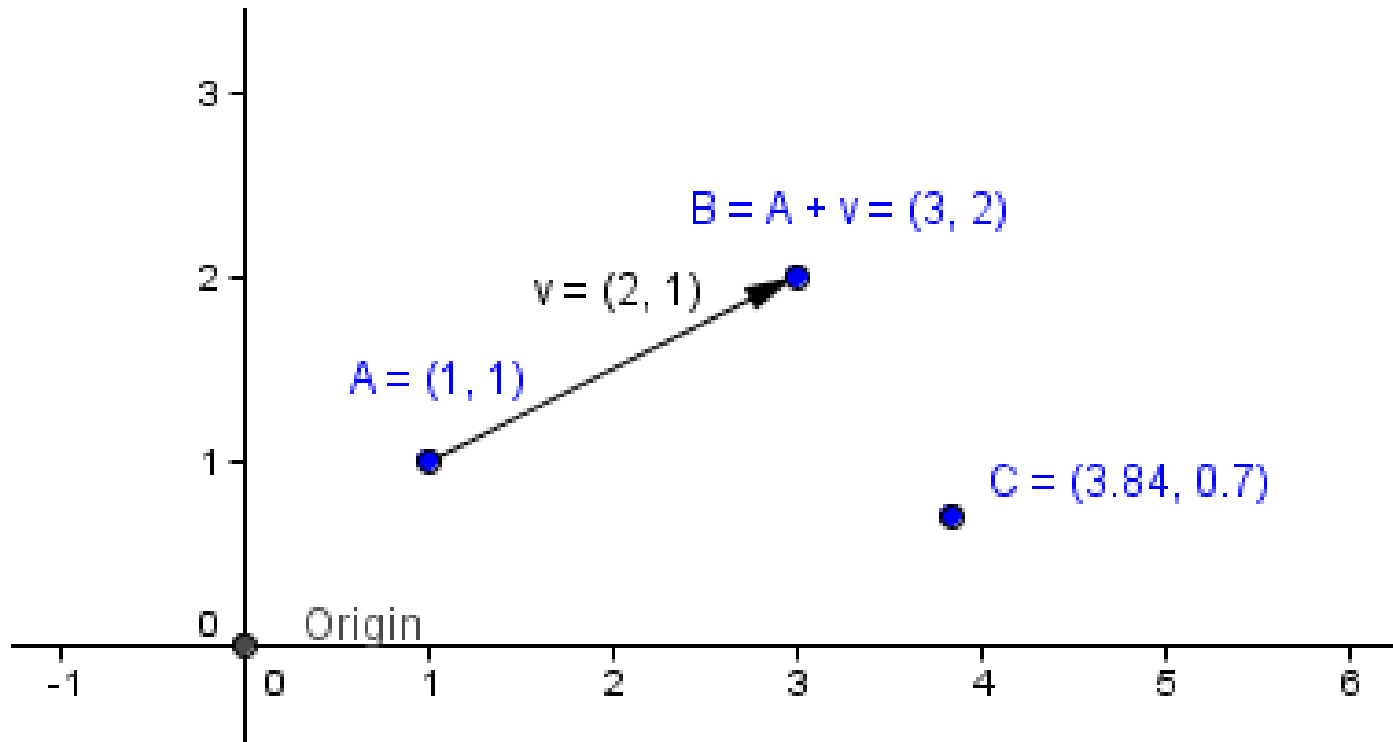
# Points and Vectors

- Besides doing operations between points, or between vectors, we want to do operations between them.
- Or do we? Can you think of an operation we would want to do between a point and a vector?



# Points and Vectors

- When we put those two vector spaces together via addition, we get an **affine space**.



# Points and Vectors

- Is this a point or a vector?

$$x = \begin{pmatrix} 3 & 2 \end{pmatrix}$$

- What about this?

$$x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

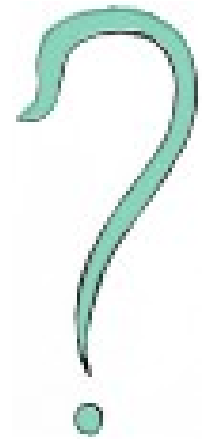


# Points and Vectors

- **Row-major** and **column-major** formats.
- Which is which?
- How to get from one to another?

$$x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$x = (3 \ 2)$$



Often you is expected to guess which one is used based on the context.

Good explanation:

<http://www.scratchapixel.com/lessons/mathematics-physics-for-computer-graphics/geometry/row-major-vs-column-major-vector>

# Points and Vectors

- **Homogeneous coordinates** – a notation where we add an additional coordinate to distinguish between points and vectors.

$$\text{2D} \quad p = (x \quad y \quad z) = \begin{pmatrix} x & y \\ z & z \end{pmatrix}$$

$$\text{3D} \quad p = (x \quad y \quad z \quad w) = \begin{pmatrix} x & y & z \\ w & w & w \end{pmatrix}$$

# Points and Vectors

- In 2D homogeneous coordinates:

- **Point**       $p = \begin{pmatrix} x & y & z \end{pmatrix} \quad z \neq 0$

- **Vector**       $p = \begin{pmatrix} x & y & z \end{pmatrix} \quad z = 0$

- Vector is a point located in infinity

# Points and Vectors

- In 3D homogeneous coordinates:

- **Point**       $p = (x \ y \ z \ w)$        $w \neq 0$

- **Vector**       $p = (x \ y \ z \ w)$        $w = 0$

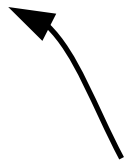
- Vector is a point located in infinity



# Points and Vectors

- What should  $z$  be if you want to define a point located at some  $(x, y)$ ?
- How does addition work now?
- Addition between two vectors?
- Addition between two points?
- Subtraction of points?

Normalized point



# Line

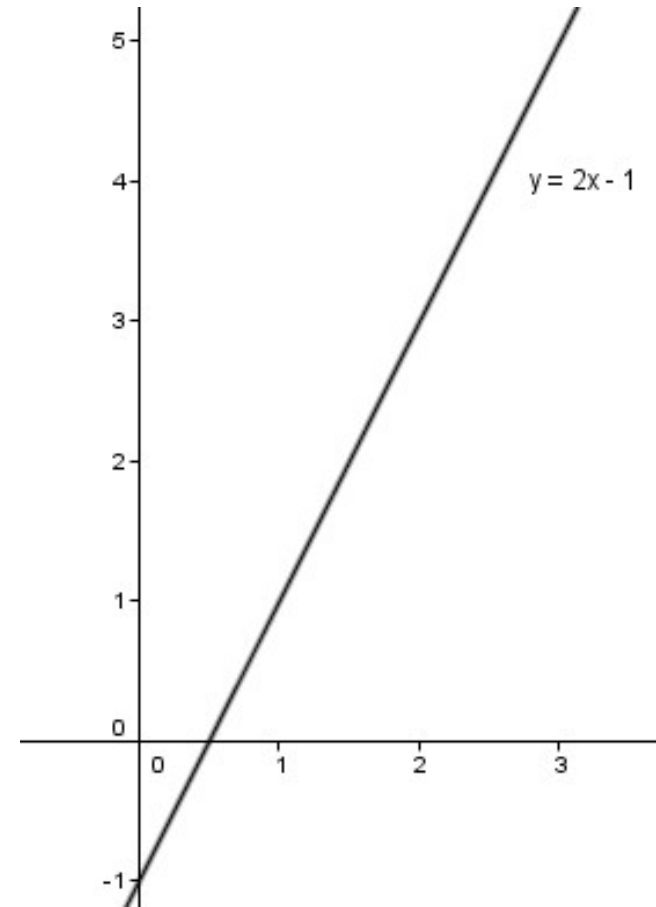
- You probably know about the **implicit** line (*sirge*)

equation:  $y = a \cdot x + b$

It defines the relationship between the ( $x$  and  $y$ ) coordinates.

- Can also be used to test if a given point is on the line or not.

How?



# Line

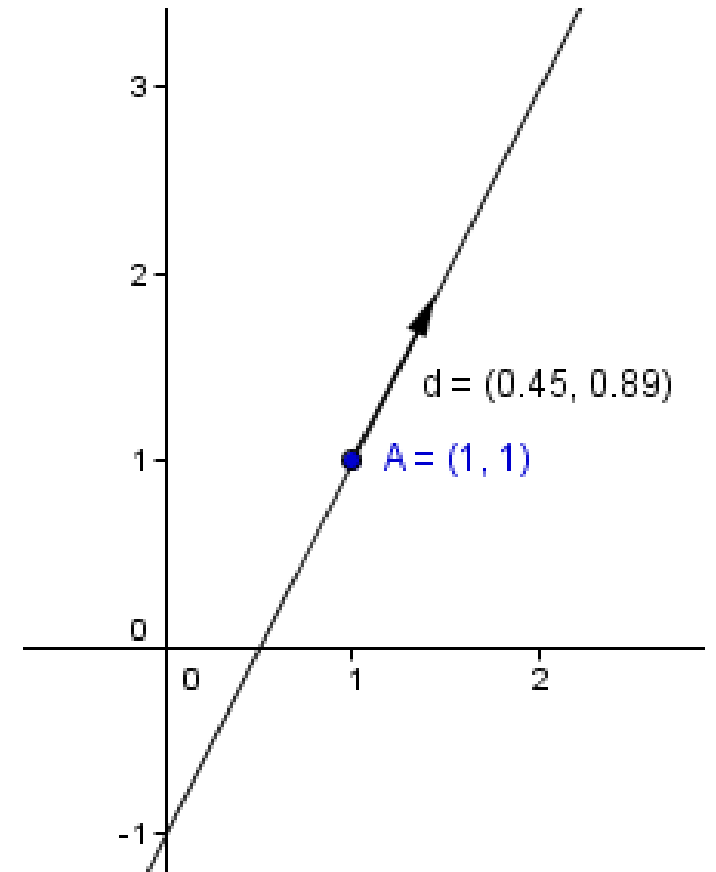
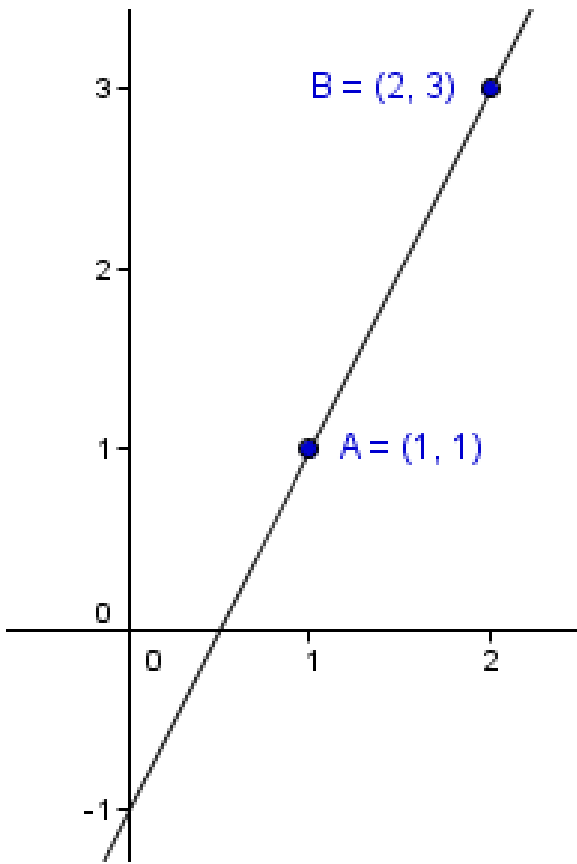
- How can we represent a **line in our affine space**?
- We do not know  $a$  (the slope) or  $b$  (y-intercept).
- What would we need to know to represent a line?



Hint: Line is a collection of points.

# Line

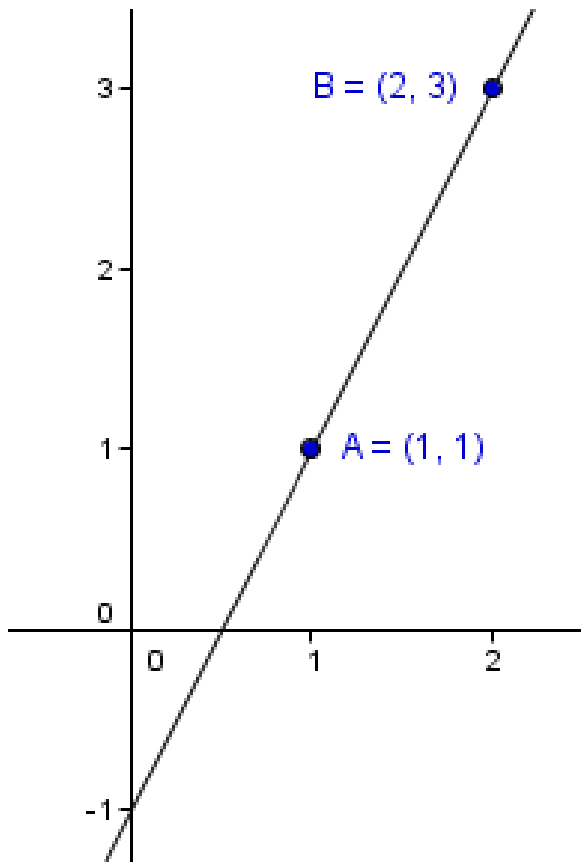
a) Two points that the line passes



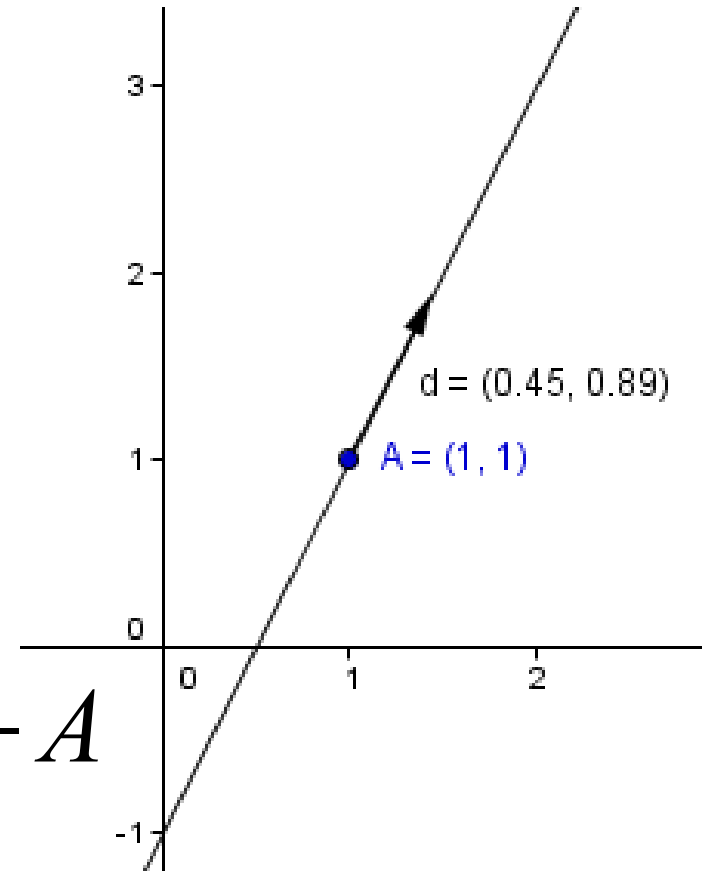
b) One point and a direction vector

# Line

$$\text{line} = (1 - \alpha) \cdot A + \alpha \cdot B$$



$$\text{line} = A + \alpha \cdot d$$

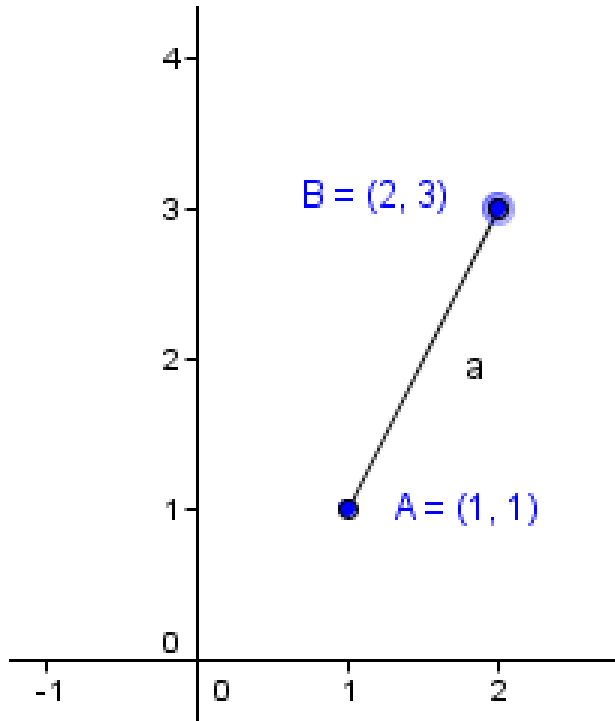


$$d = B - A$$

# Line Segment

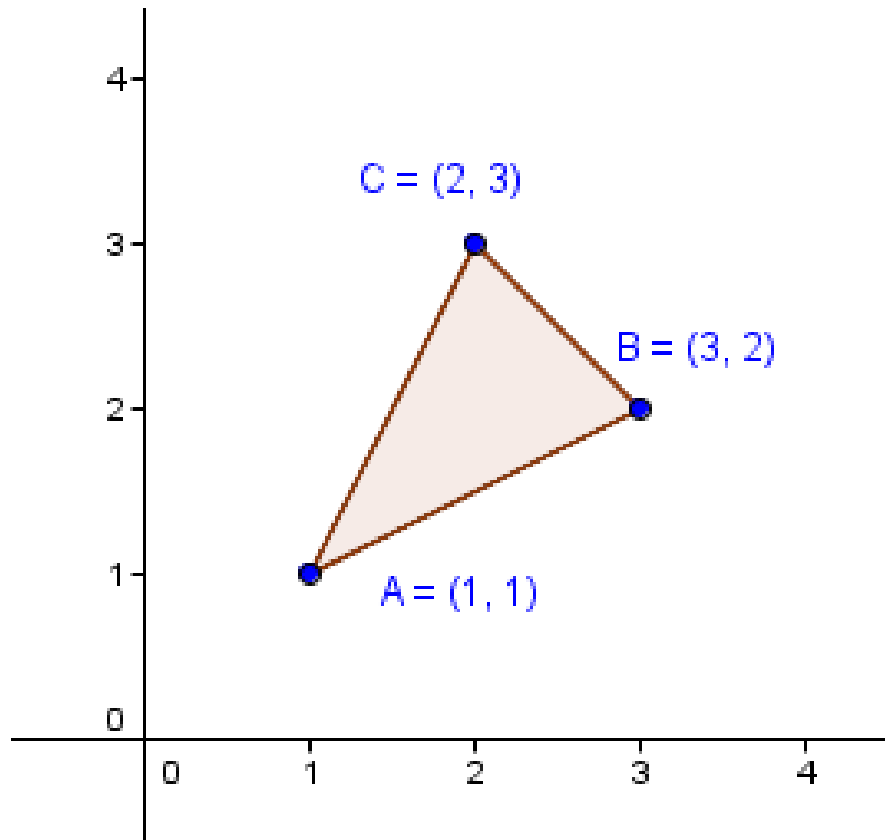
- Knowing that:  $line = (1 - \alpha) \cdot A + \alpha \cdot B$

How to represent a line segment (*sirglōik*)?



# Triangle

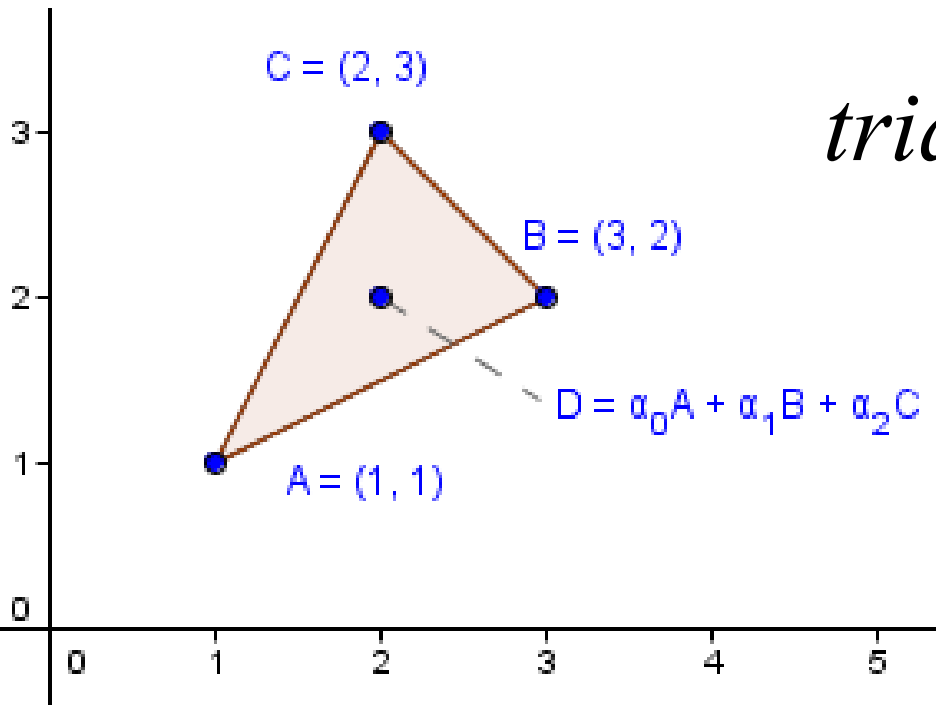
- How about a triangle?



# Barycentric Coordinates

(*kumer kombinatsioon*)

- The coefficients of a **convex combination** of the vertices are the **Barycentric coordinates** of points in the triangle.



$$\text{triangle} = \alpha_0 \cdot A + \alpha_1 \cdot B + \alpha_2 \cdot C$$

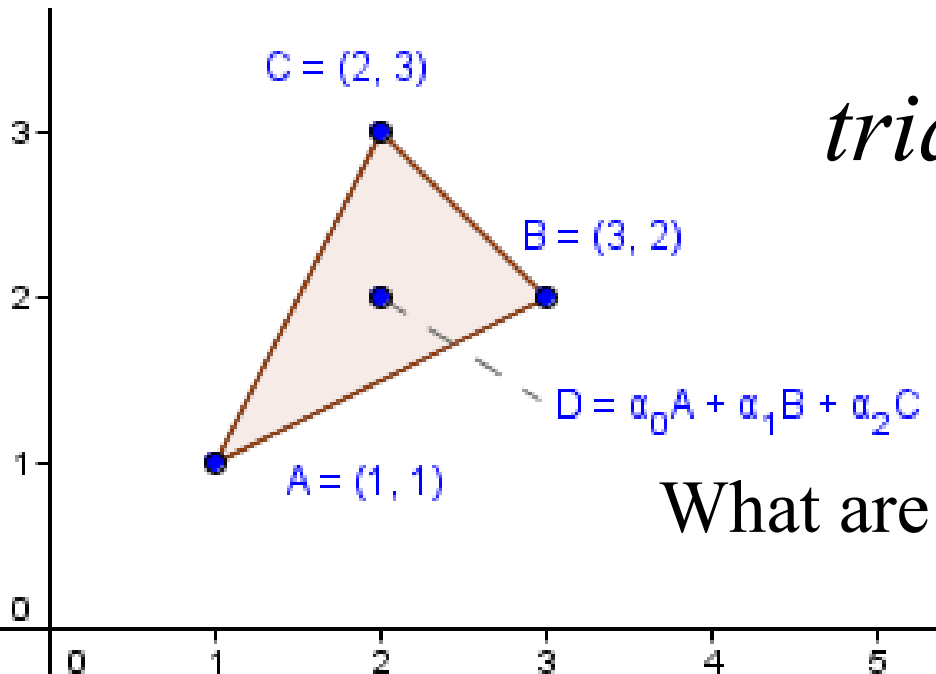
$$\alpha_i \geq 0, \quad \alpha_0 + \alpha_1 + \alpha_2 = 1$$



# Barycentric Coordinates

(*kumer kombinatsioon*)

- The coefficients of a **convex combination** of the vertices are the **Barycentric coordinates** of points in the triangle.



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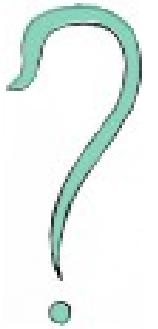
What are the coordinates of the vertices in the Barycentric system?

Find them for other easy points.



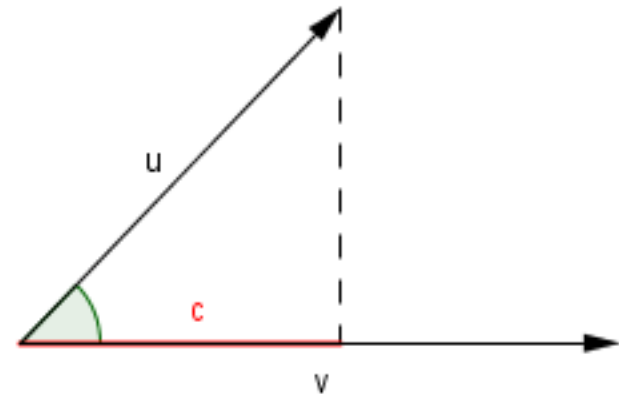
# Dot Product

- Useful operation between vectors. Why?
- Definition
  - Geometric:  $u \cdot v = \|u\| \cdot \|v\| \cdot \cos(\angle uv)$
  - Algebraic:  $u \cdot v = u_0 \cdot v_0 + u_1 \cdot v_1 + u_2 \cdot v_2$
- Also called: scalar product, inner product
- *Skalaarkorrutis*



# Scalar Projection

- Dot product allows to project one vector onto another.
- Scalar projection of  $u$  onto  $v$  is: 
$$c = u \cdot \frac{v}{\|v\|} = u \cdot \hat{v}$$
- It gives you the length, how much  $\hat{v}$  you have to take to reach the orthogonal projection point of  $u$ .



*Eg* The Gram–Schmidt process

# Cross Product

- Returns a vector orthogonal to the operands.

- Definition  $u \times v = n \cdot \|u\| \cdot \|v\| \cdot \sin(\angle uv)$

- Geometric

- Algebraic

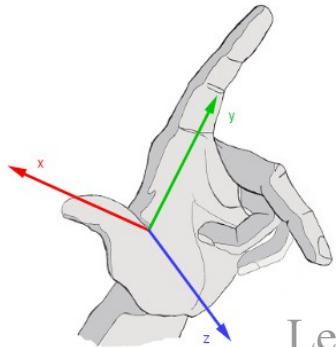
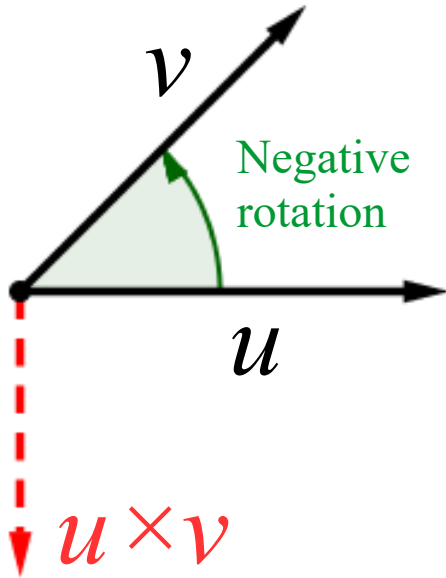
- Also called: vector product

- *Vektorkorrrutis*

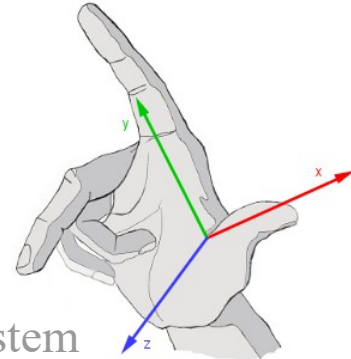
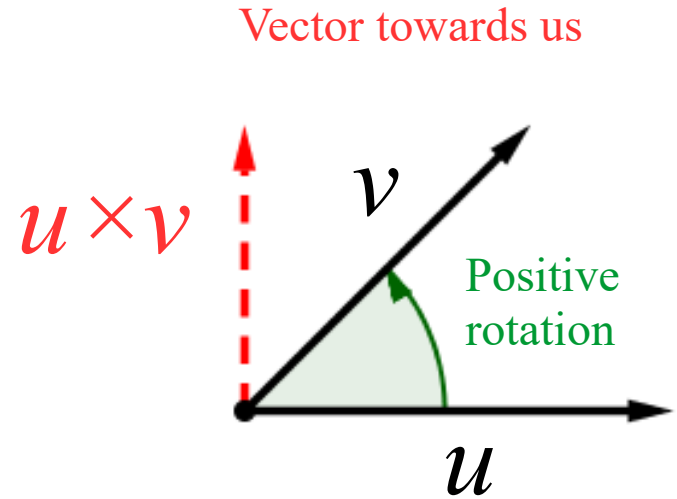
$$u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

Direction of the result depends on the handedness of the coordinate system.

# Cross Product



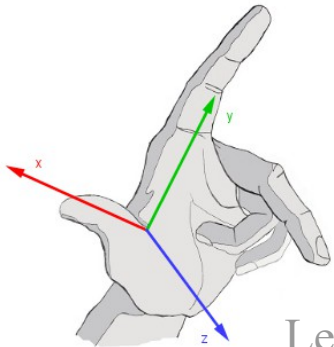
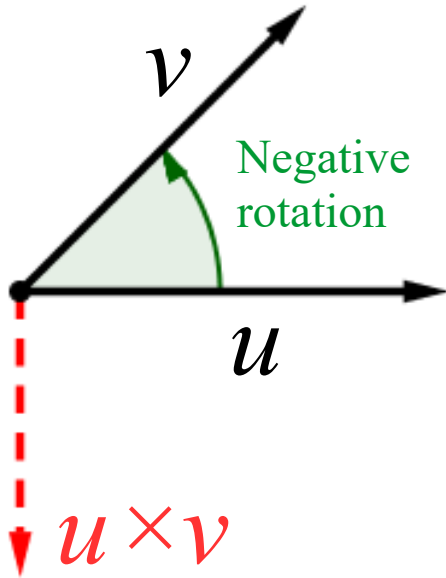
Left-handed system



Right-handed system

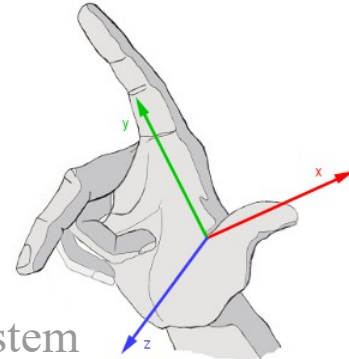
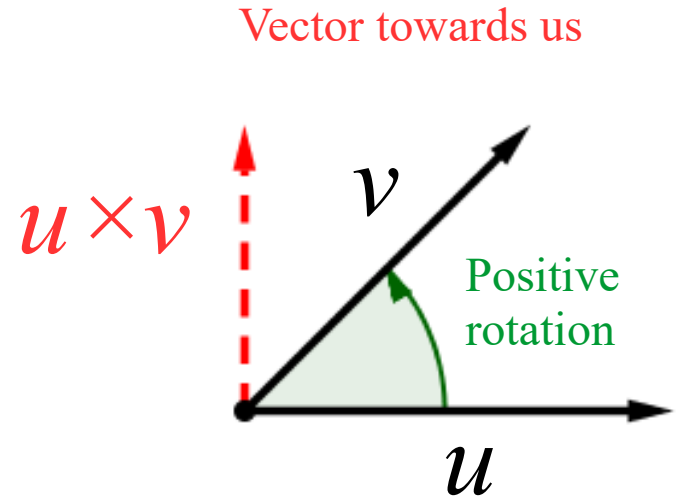
# In which world are we living in?

(think bottlecaps and screws)



Left-handed system

Vector away from us



Right-handed system

Vector towards us

# Scalar Triple Product

- Definition:  $u \cdot (v \times w)$
- Useful in solving a system of equations of vectors.

$$u \cdot (v \times w) = \begin{vmatrix} u_0 & u_1 & u_2 \\ v_0 & v_1 & v_2 \\ w_0 & w_1 & w_2 \end{vmatrix}$$

- We see this in Basic II, with triangle-ray intersections.
- *Segakorrutis*.

What was important for you today?

What more would you like to know?

Next time: Transformations  
(scale, shear, rotate, translate)