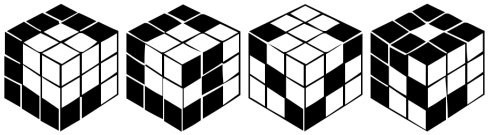
# Laplacian Growth

Mathias Plans

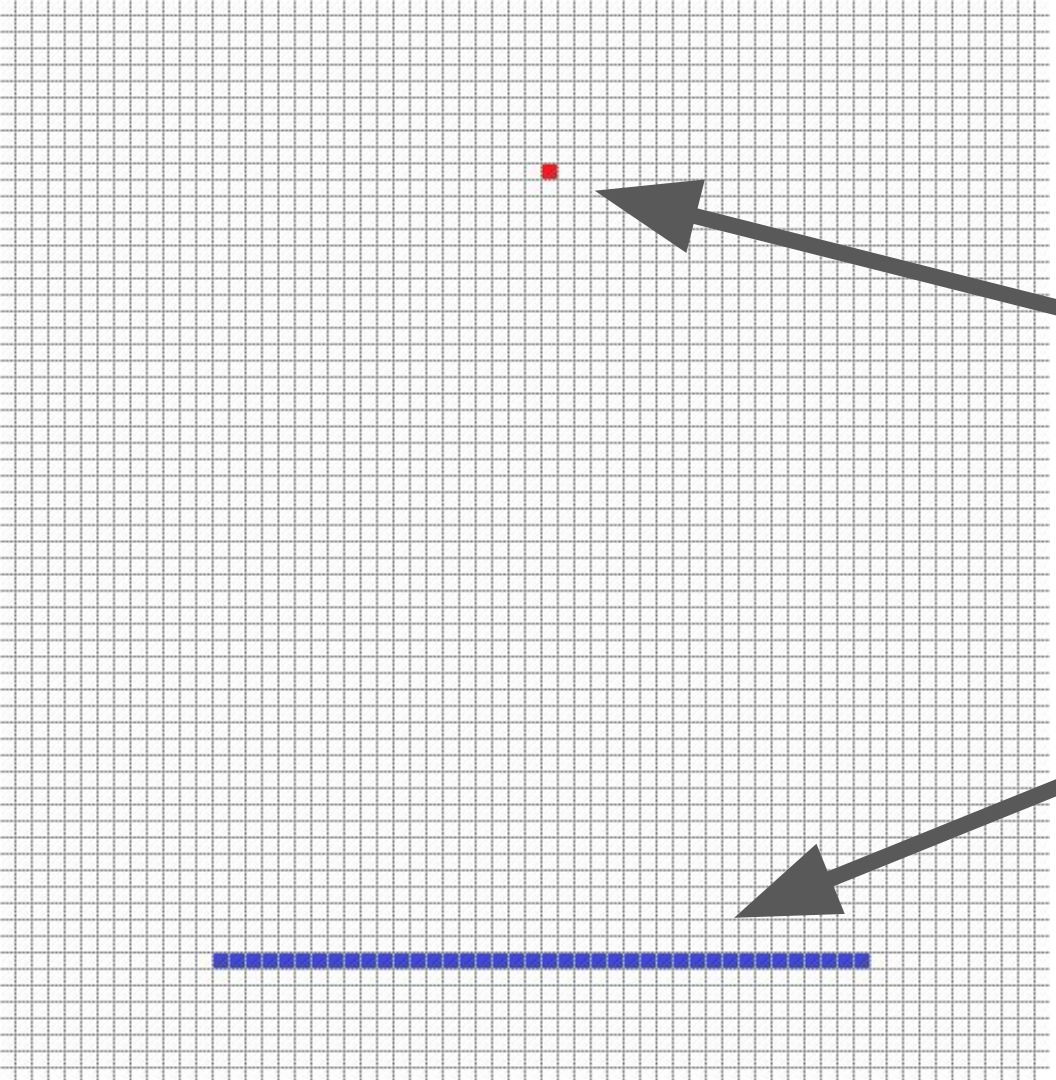


# What is to come

- Laplacian growth
- Laplacian operator
- Graph Laplacian
- Computation
- Use cases

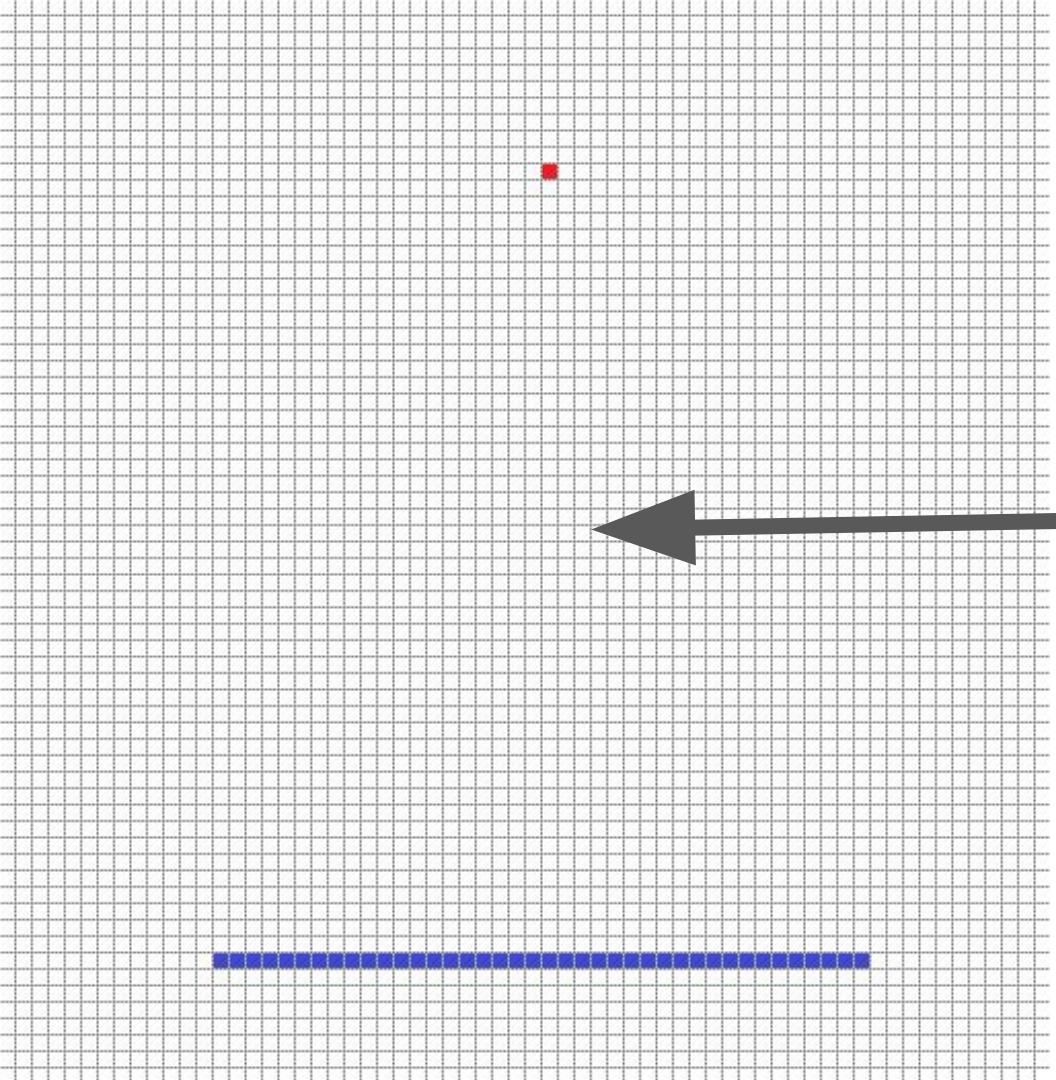
# Laplacian field

$$\Phi$$

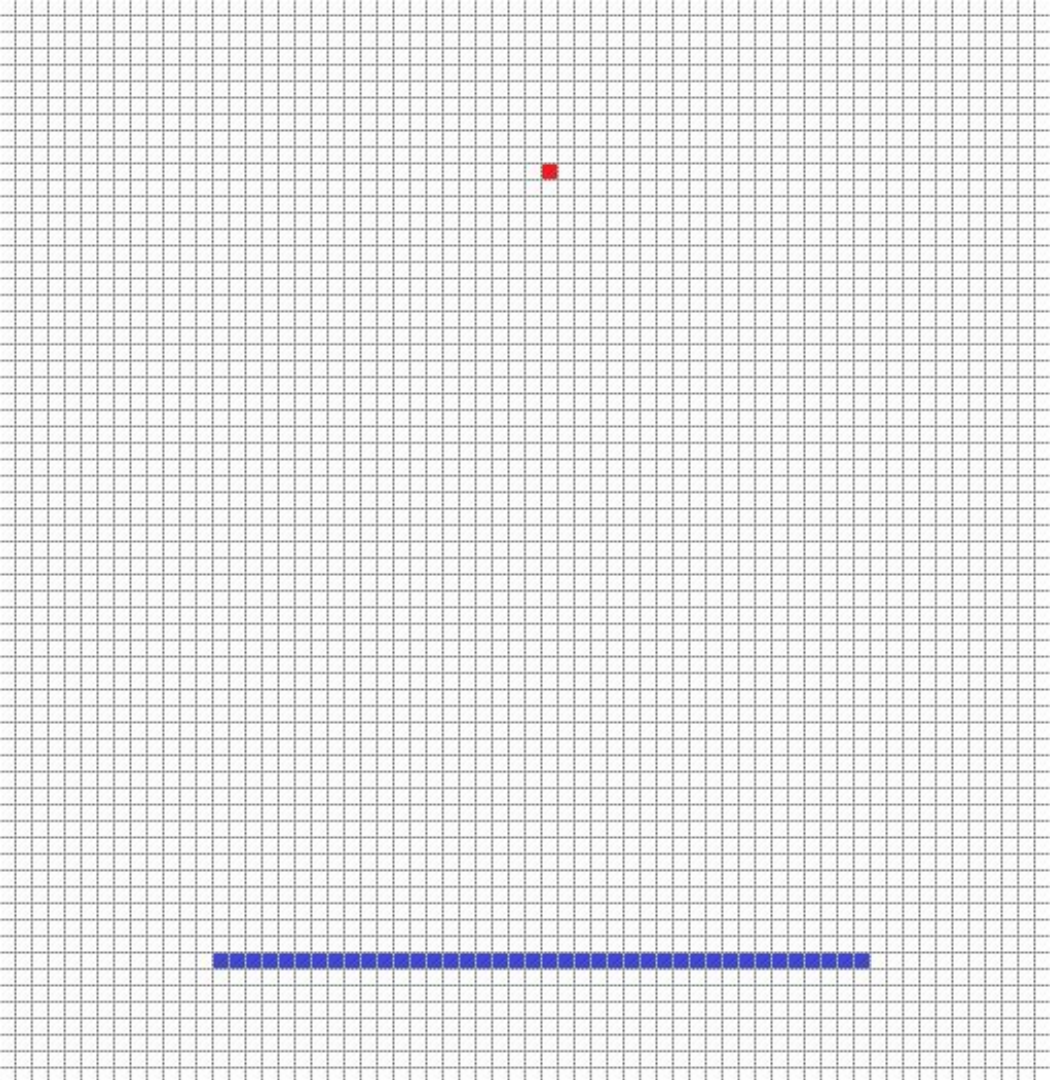


$$\Phi = 0$$

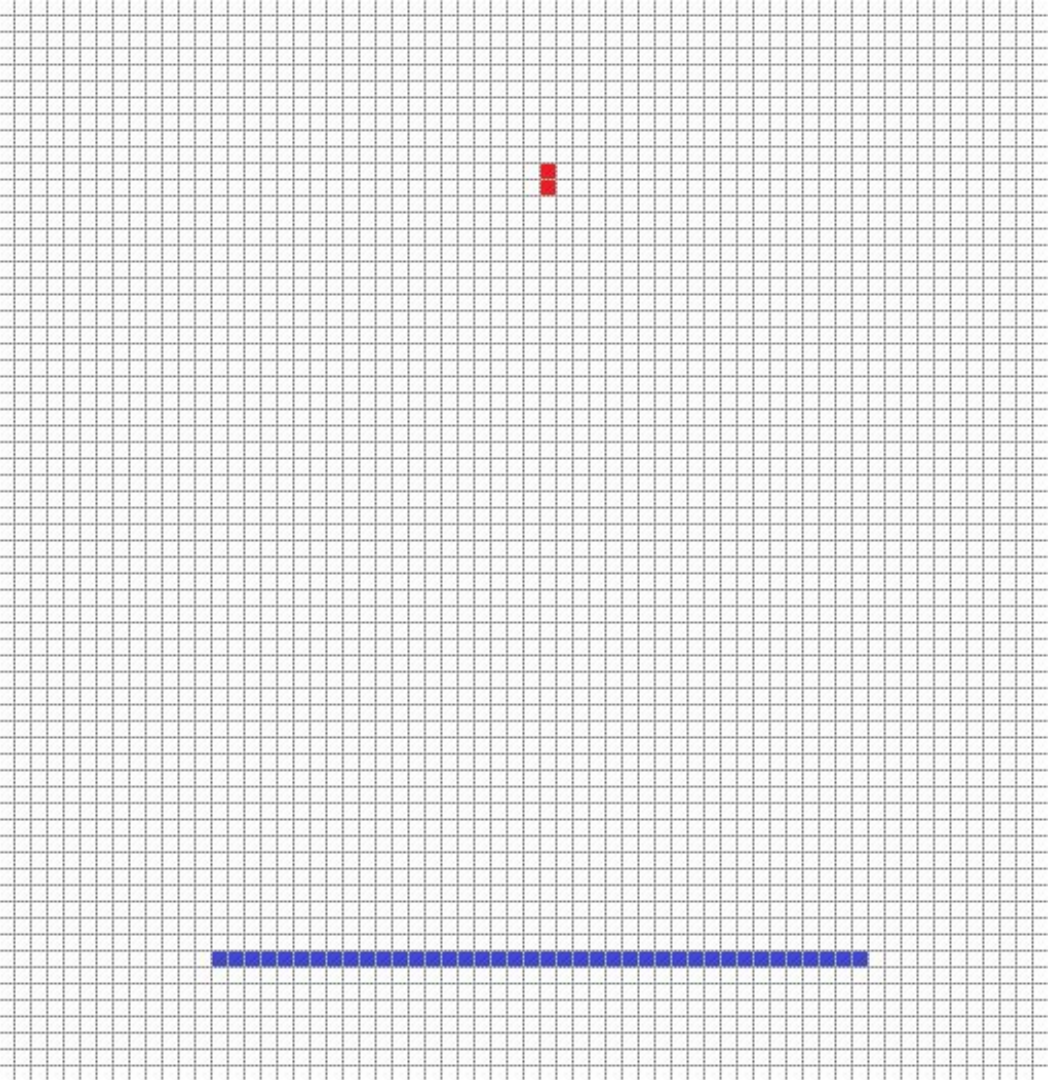
$$\Phi = 1$$

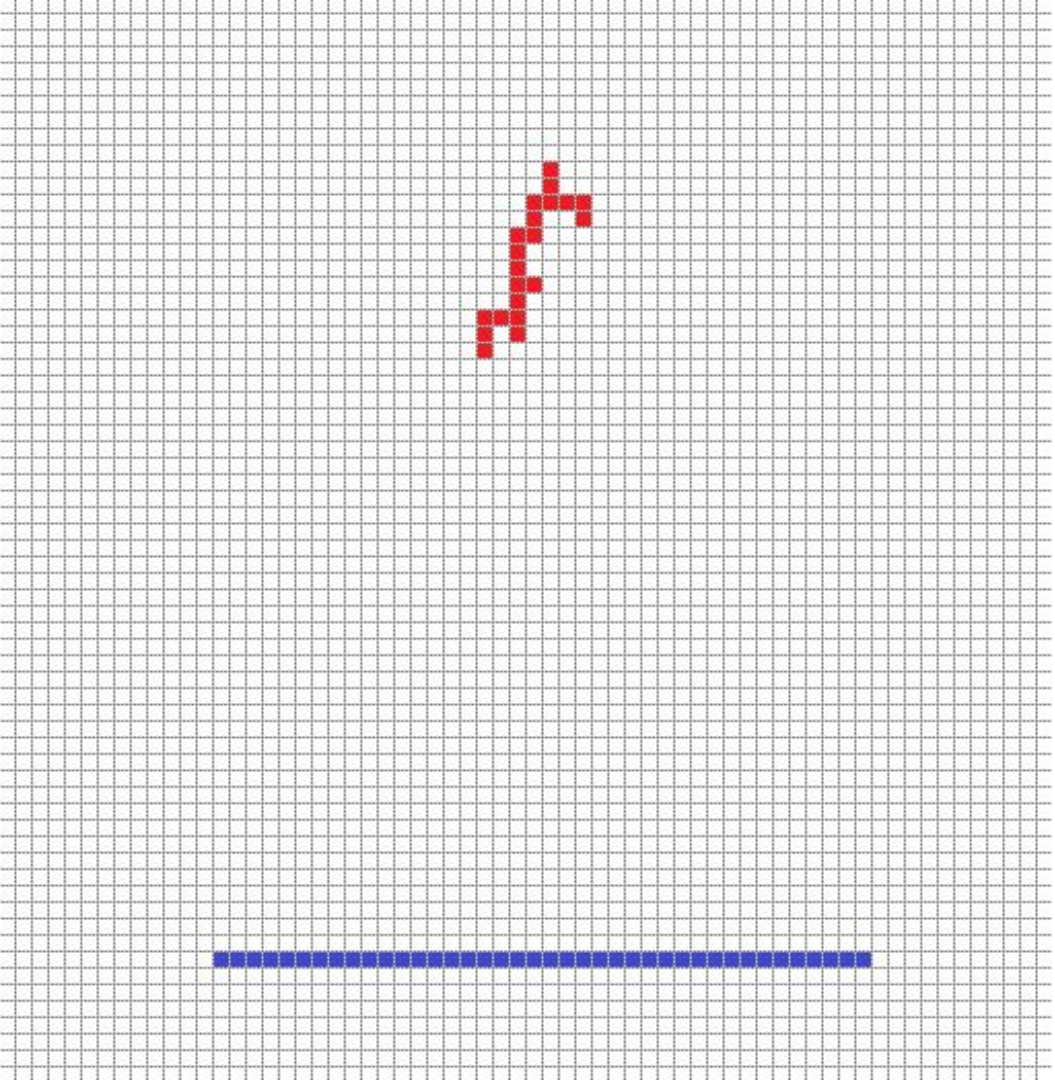


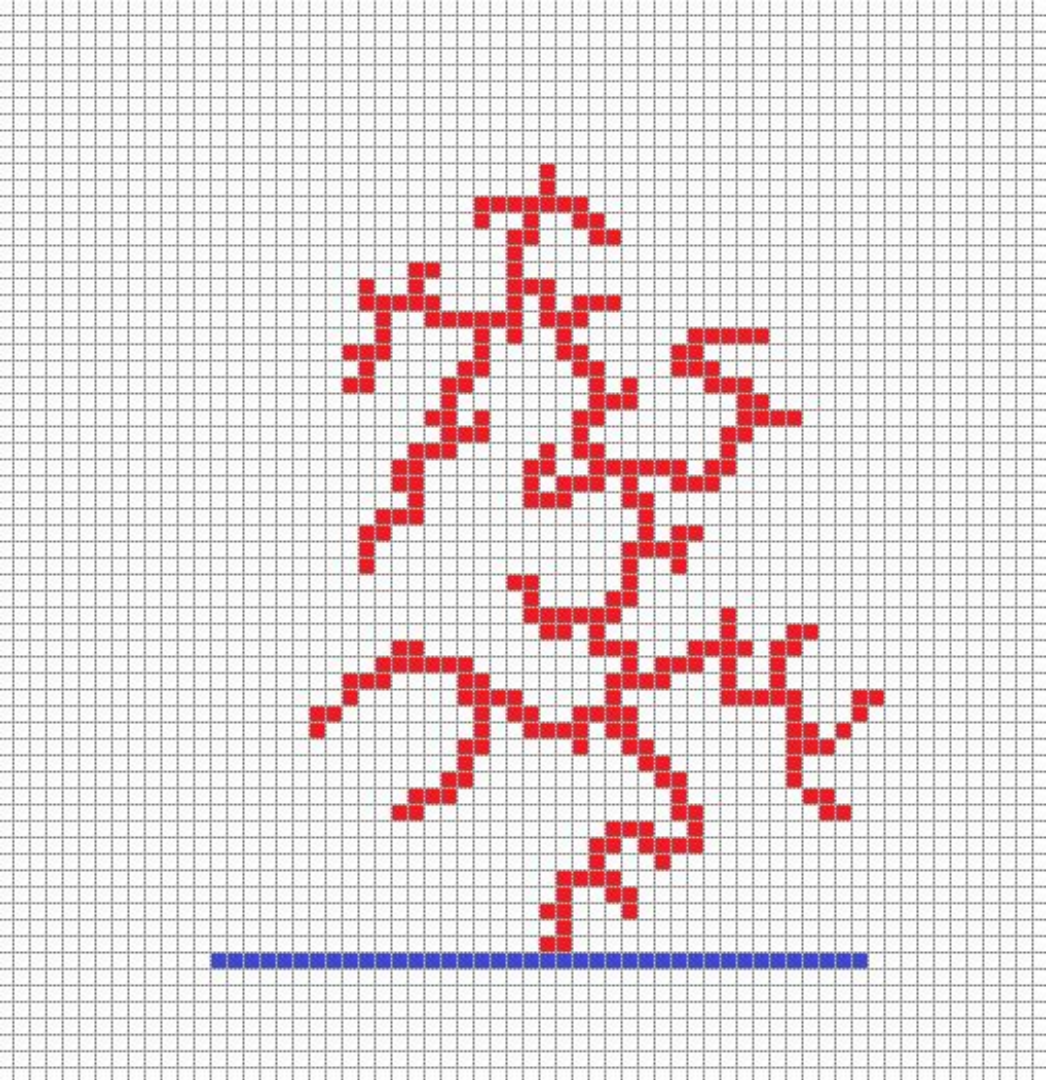
$$\Phi = ?$$

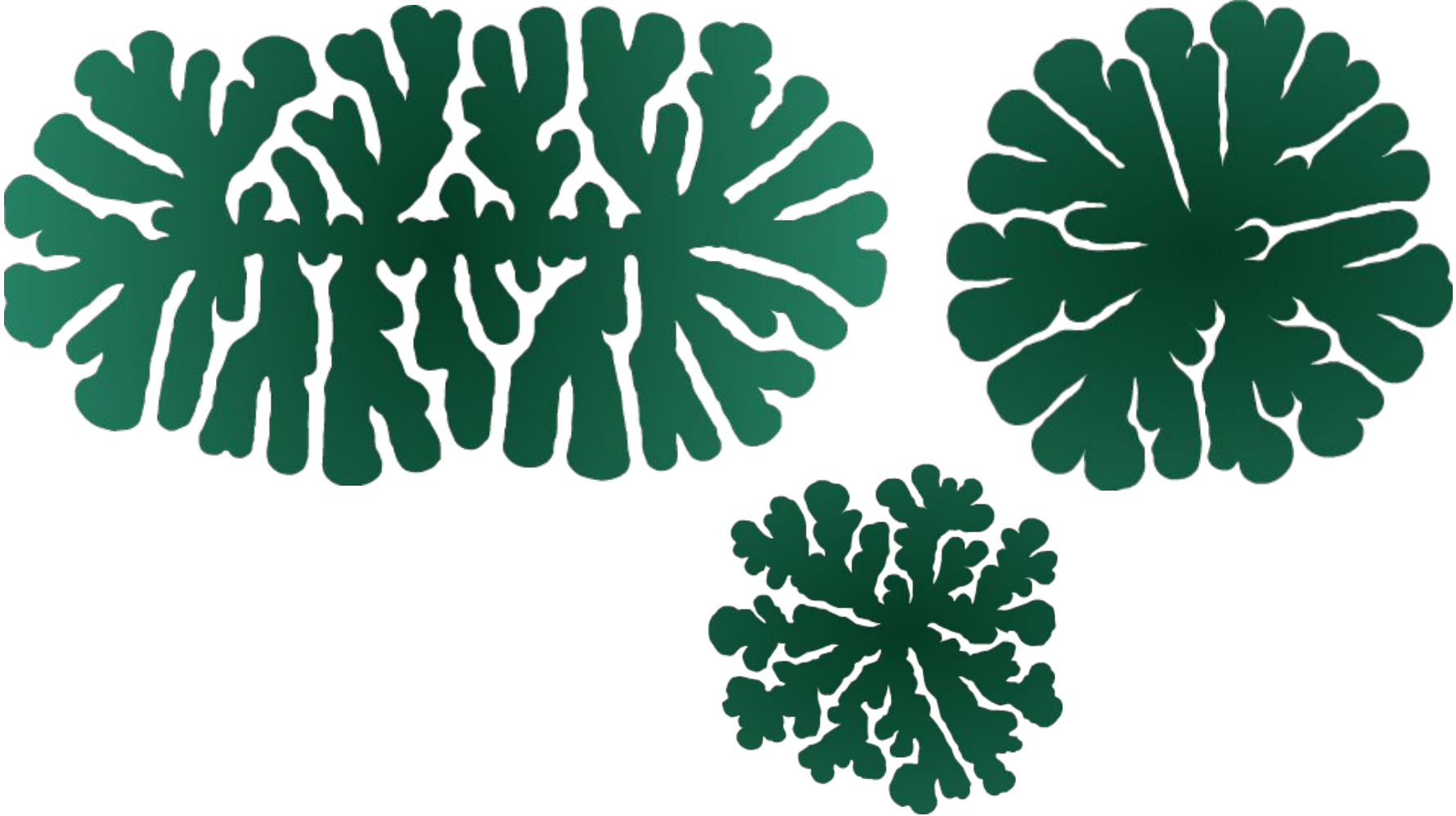


$$\nabla^2 \Phi = 0$$

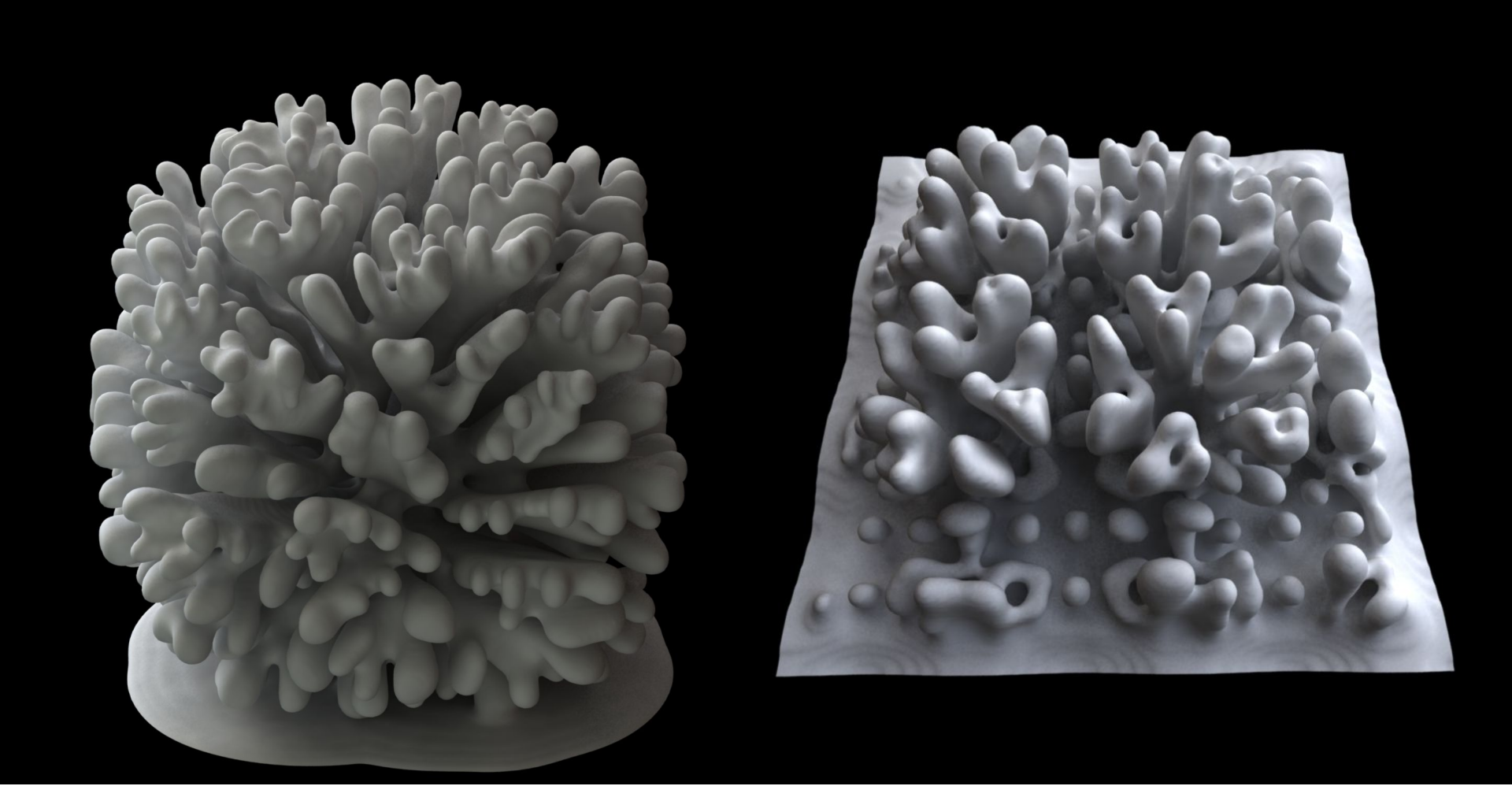








https://pixelero.wordpress.com/2011/11/18/laplacian-growth-vector-graphics/



https://n-e-r-v-o-u-s.com/blog/?p=1536

https://unsplash.com/photos/0G01UI1MQhg

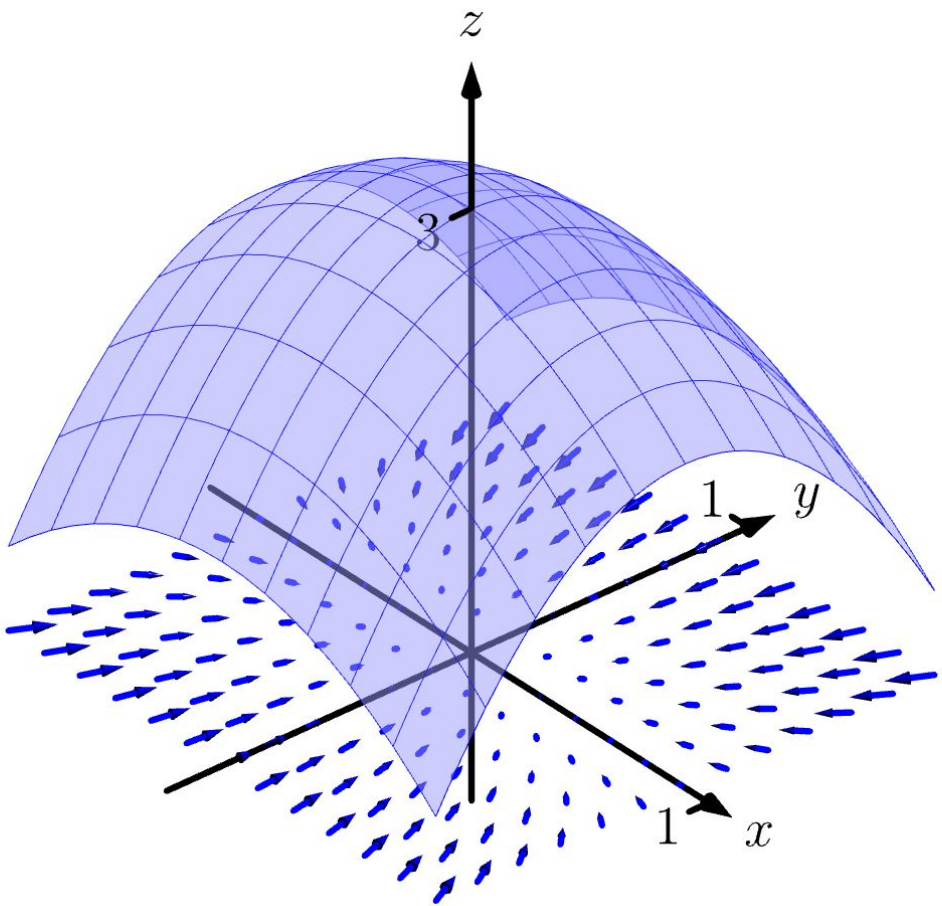

https://www.peridotfinejewelry.com/products/jamie-joseph-smooth-round-dendritic-agate-ring

https://www.farwayart.com/shop/scenic-dendrite-agate

https://stormandsky.com/lightning

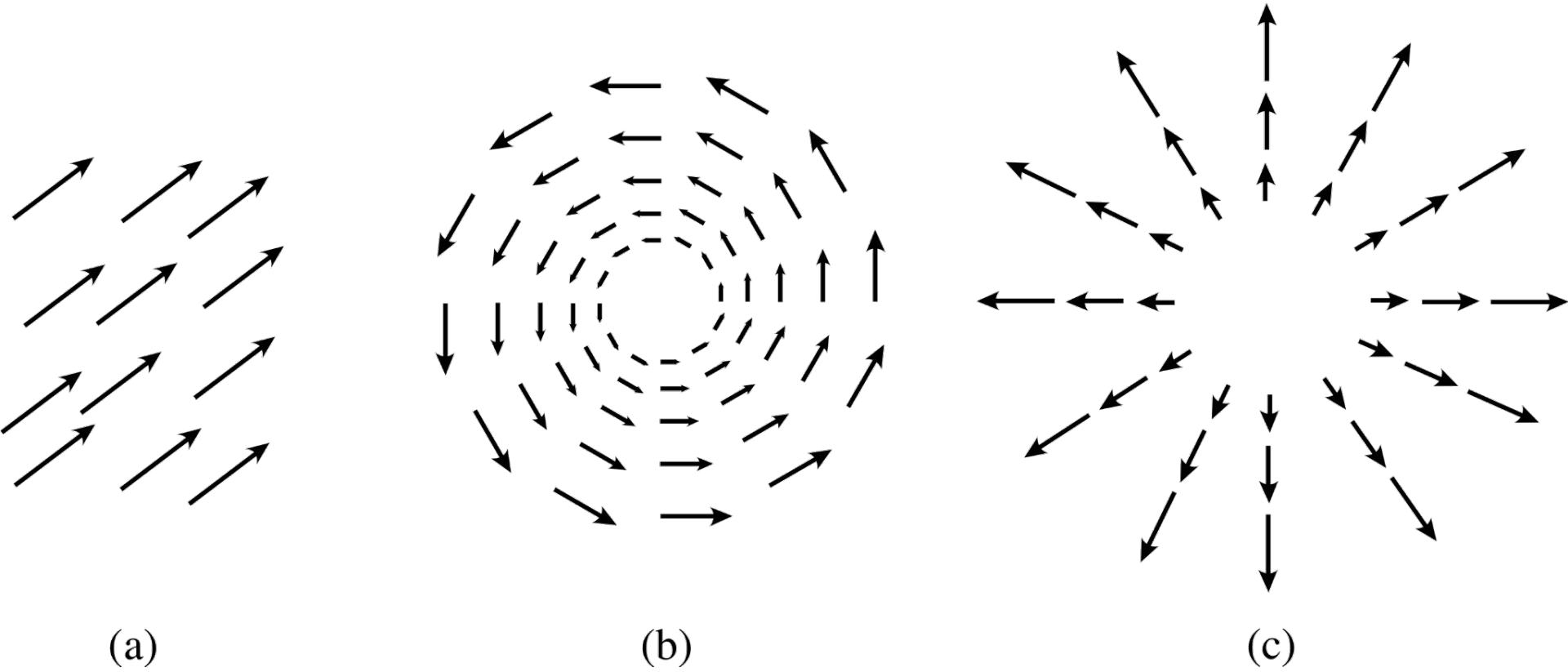
# Maths

$$\partial$$



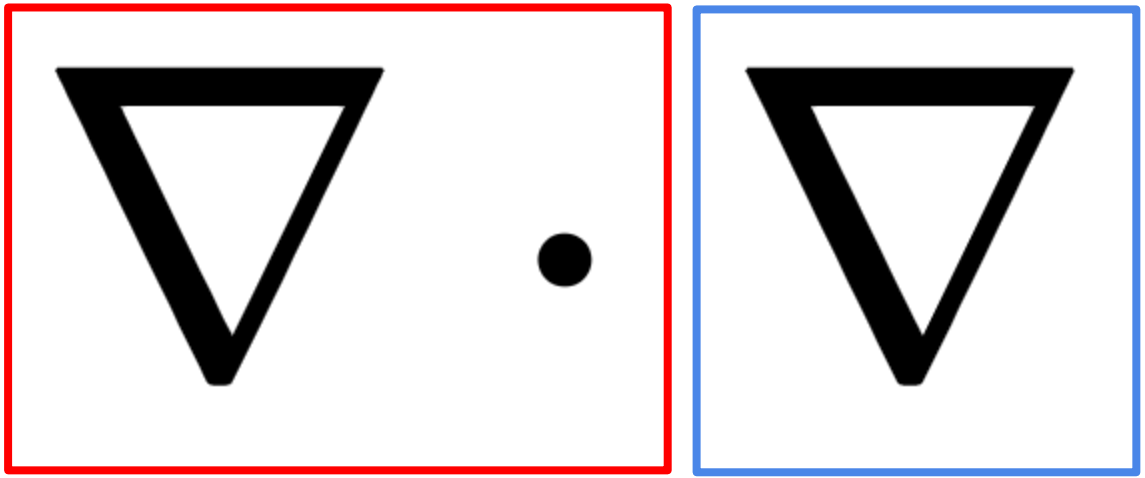

https://sites.und.edu/timothy.prescott/apex/web/apex.Ch15.S2.html

$$\nabla \cdot$$




https://favpng.com/png_view/sample-vector-curl-divergence-vector-field-vector-calculus-png/YwAaqMfb#_=_

$$\nabla \cdot \boxed{\nabla}$$

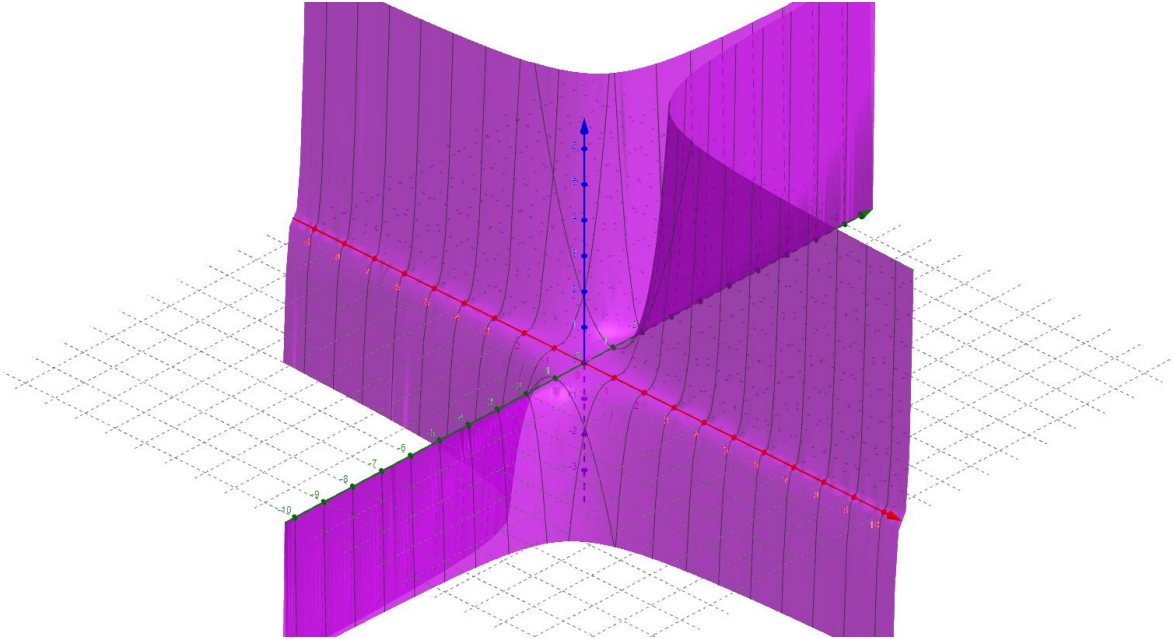


$$\nabla^2$$

$$\nabla^2 = \Delta$$

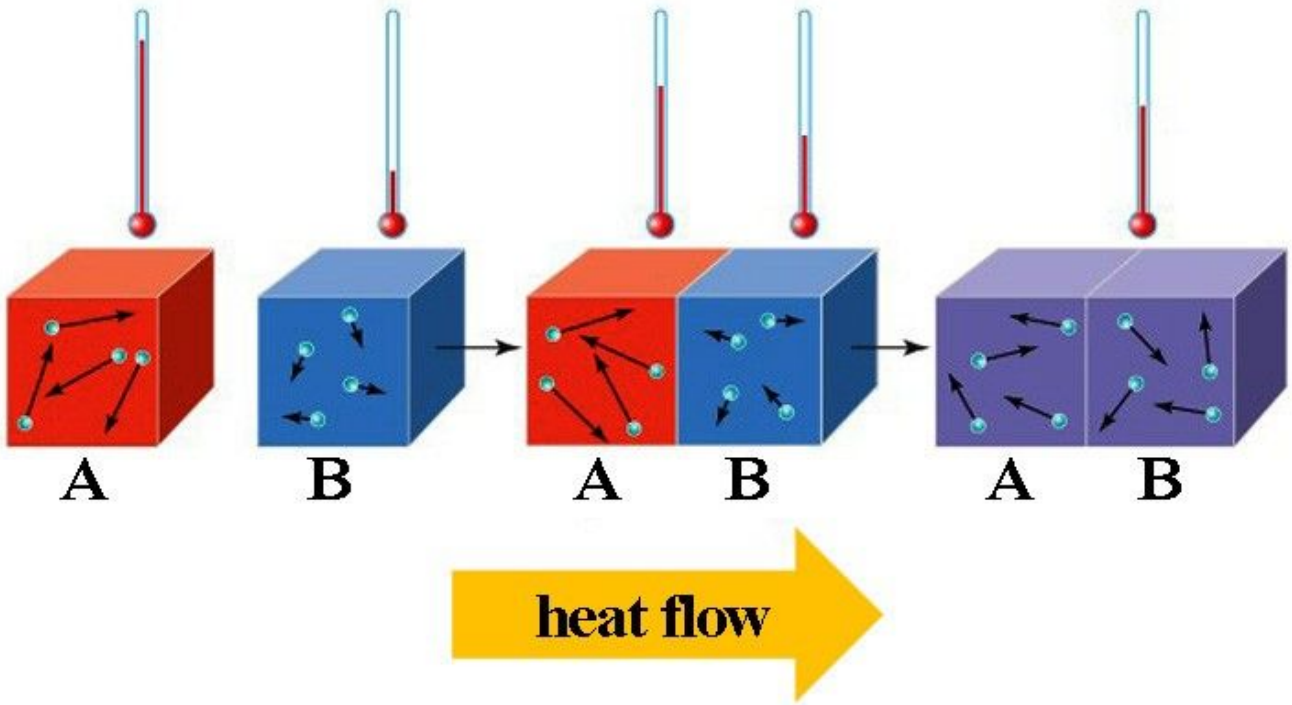
$$f''$$

$$f(x, y) = x^2 y^3$$



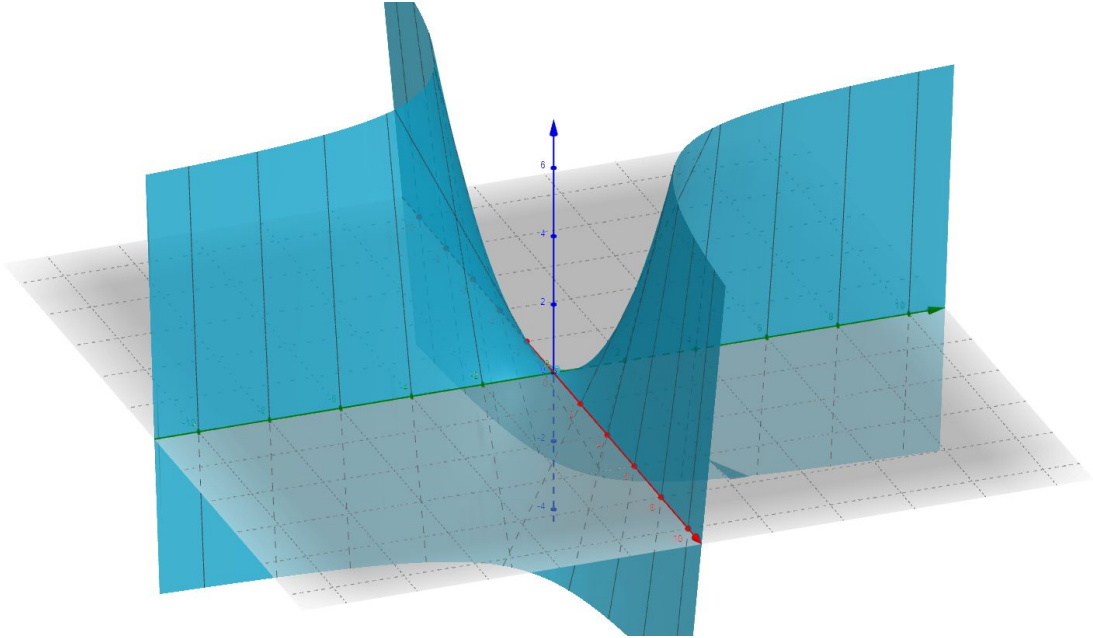
GeoGebra

$$\nabla^2 f = 0$$



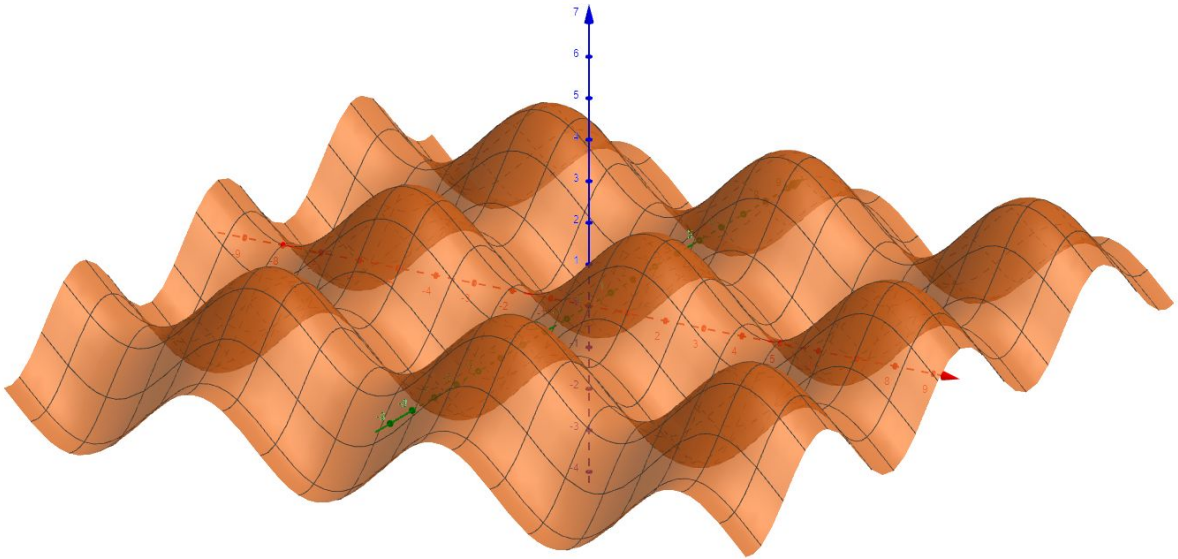

https://www.toppr.com/ask/content/story/amp/thermal-equilibrium-and-the-zeroth-law-of-thermodynamics-64460/

$$f(x, y) = xy$$



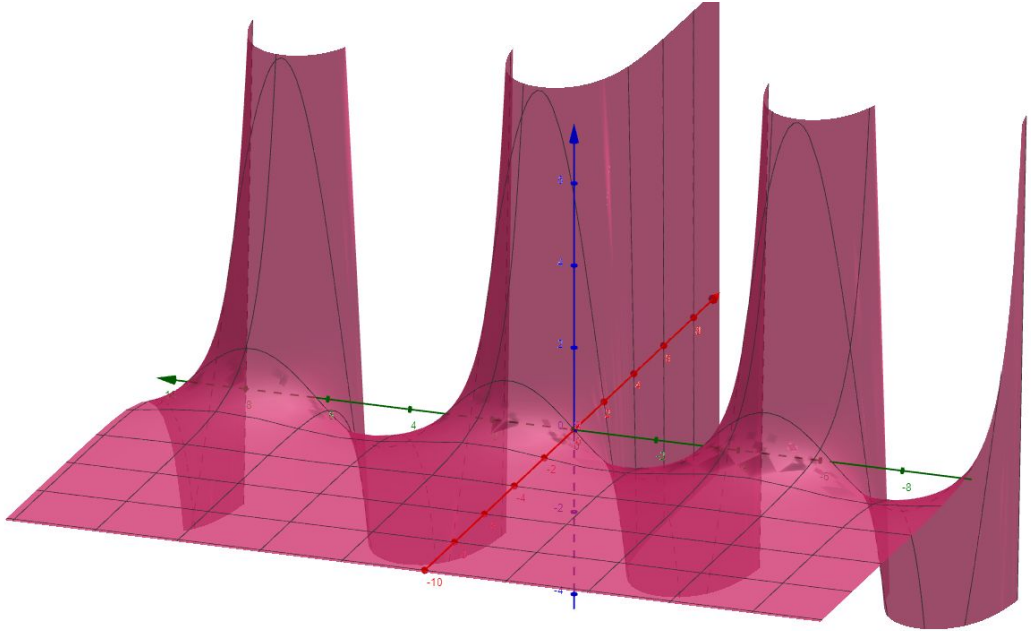
GeoGebra

$$f(x, y) = sin\ x + cos\ y$$



GeoGebra

$$f(x, y) = e^x \sin y$$

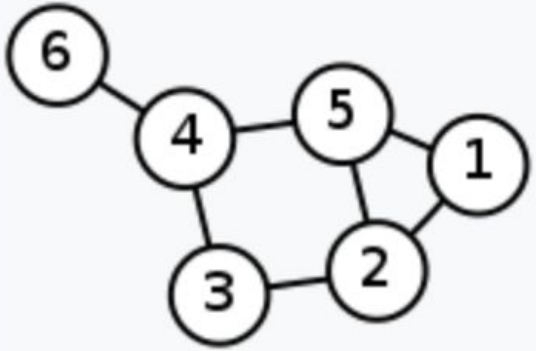


GeoGebra

# Discretization

$$\nabla^2 \phi(v) = \sum_{w:d(w,v)=1} \phi(v) - \phi(w)$$

https://en.wikipedia.org/wiki/Laplacian_matrix

| Labelled graph | Degree matrix |
|---|---|
|   | $\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ |

| Adjacency matrix | Laplacian matrix |
|---|---|
| $\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$ |

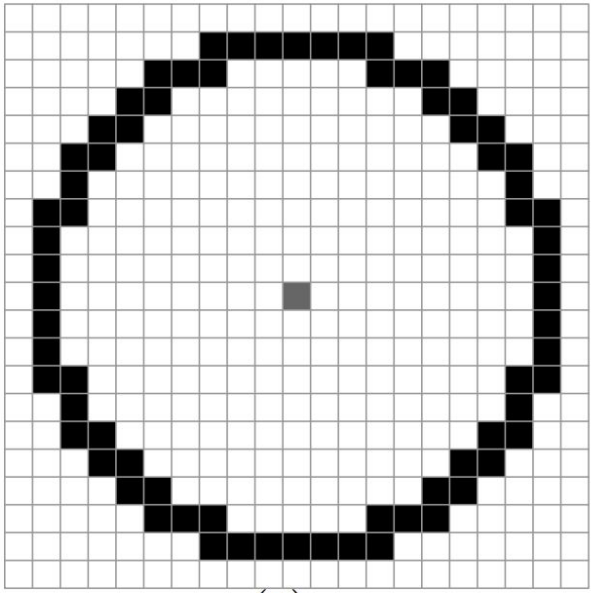
$$\nabla^2 \phi = 0$$

# Back to the Laplacian growth

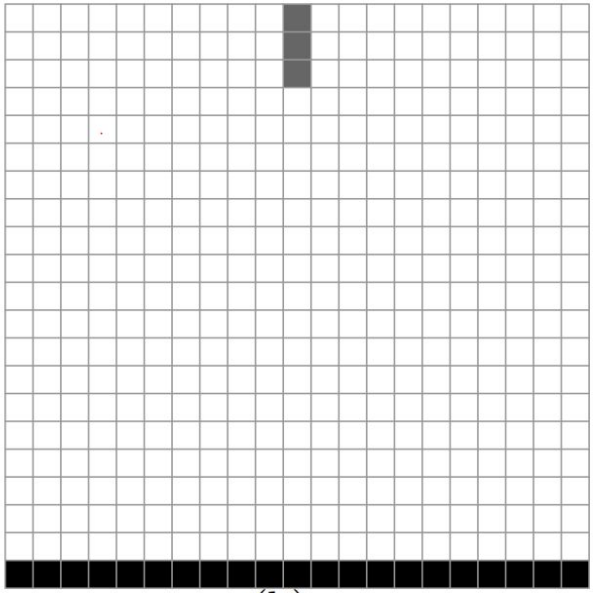
1. Set the initial values
2. Find a harmonic function
3. Select growth locations based on the harmonic function
4. Repeat from step 2 until the growth is desirable
5. Post-process

# 1. Set the initial values

- Depends on the scene
- ..and the problem



(a)



(b)

https://gamma.cs.unc.edu/FAST_LIGHTNING/lightning_tvcg_2007.pdf

2. Find a harmonic function

$$\nabla^2 \phi = 0$$

# 2. Find a harmonic function

- System of linear equations
- When $n$ is the number of vertices in the graph
  - Gaussian has time complexity of $O(n^3)$
  - Conjugate Gradient and Successive Overrelaxation have time complexity of $O(n^{1.5})$ but are more specialized
  - Multigrid method has time complexity of $O(n)$

$$\nabla^2 \phi = 0$$

| Algorithm | Type | Serial Time | PRAM Time | Storage | #Procs |
|---|---|---|---|---|---|
| Dense LU | D | N^3 | N | N^2 | N^2 |
| Band LU | D | N^2 | N | N^(3/2) | N |
| Inv(P)*bhat | D | N^2 | log N | N^2 | N^2 |
| Jacobi | I | N^2 | N | N | N |
| Sparse LU | D | N^(3/2) | N^(1/2) | N*log N | N |
| CG | I | N^(3/2) | N^(1/2)*log N | N | N |
| SOR | I | N^(3/2) | N^(1/2) | N | N |
| FFT | D | N*log N | log N | N | N |
| Multigrid | I | N | (log N)^2 | N | N |
| Lower Bound | | N | log N | N | |

https://people.eecs.berkeley.edu/~demmel/cs267/lecture24/lecture24.html

LU is Gaussian elimination

# Parallelized

| | # flops | # messages | # words sent |
|---|---|---|---|
| SOR | N^(3/2)/p | N^(1/2) | N/p |
| FFT | N*log(N)/p | p^(1/2) | N/p |
| Multigrid | N/p + log(p)*log(N) | (log(N))^2 | (N/p)^(1/2) + log(p)*log(N) |

https://people.eecs.berkeley.edu/~demmel/cs267/lecture25/lecture25.html

p is the number of processes

# Reducing $n$

- $n$ is usually very large
- 3D grid of size 1000 has 1 billion vertices
- Can reduce granularity

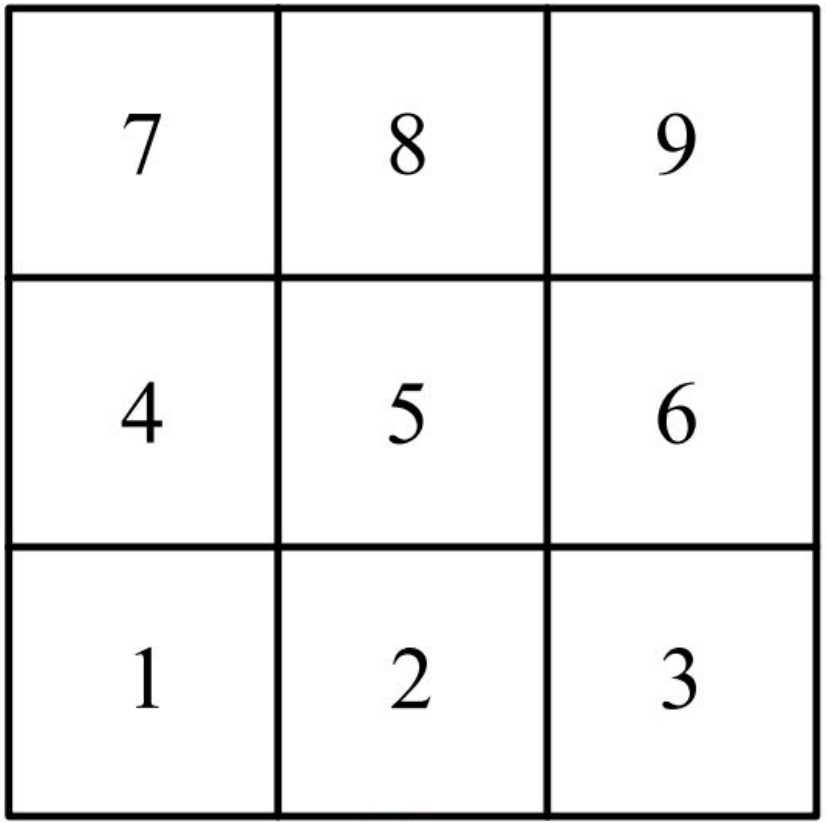
# Reducing $n$

- $n$ is usually very large
- 3D grid of size 1000 has 1 billion vertices
- Can reduce granularity
- Or..

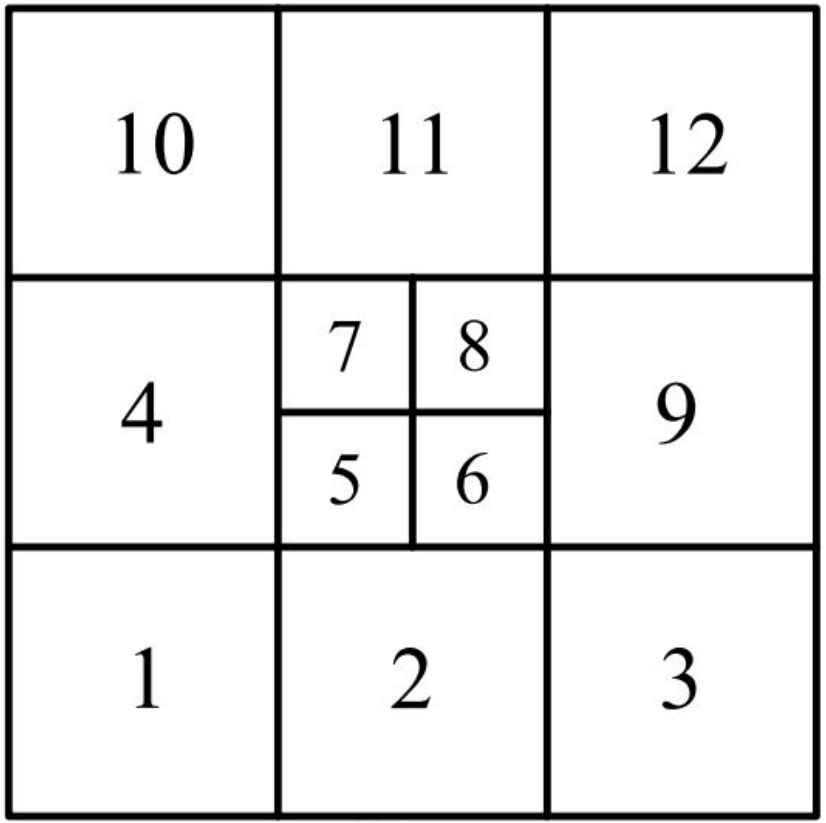
# Fast Animation of Lightning Using An Adaptive Mesh

Theodore Kim and Ming C. Lin
University of North Carolina at Chapel Hill

https://gamma.cs.unc.edu/FAST_LIGHTNING/lightning_tvcg_2007.pdf

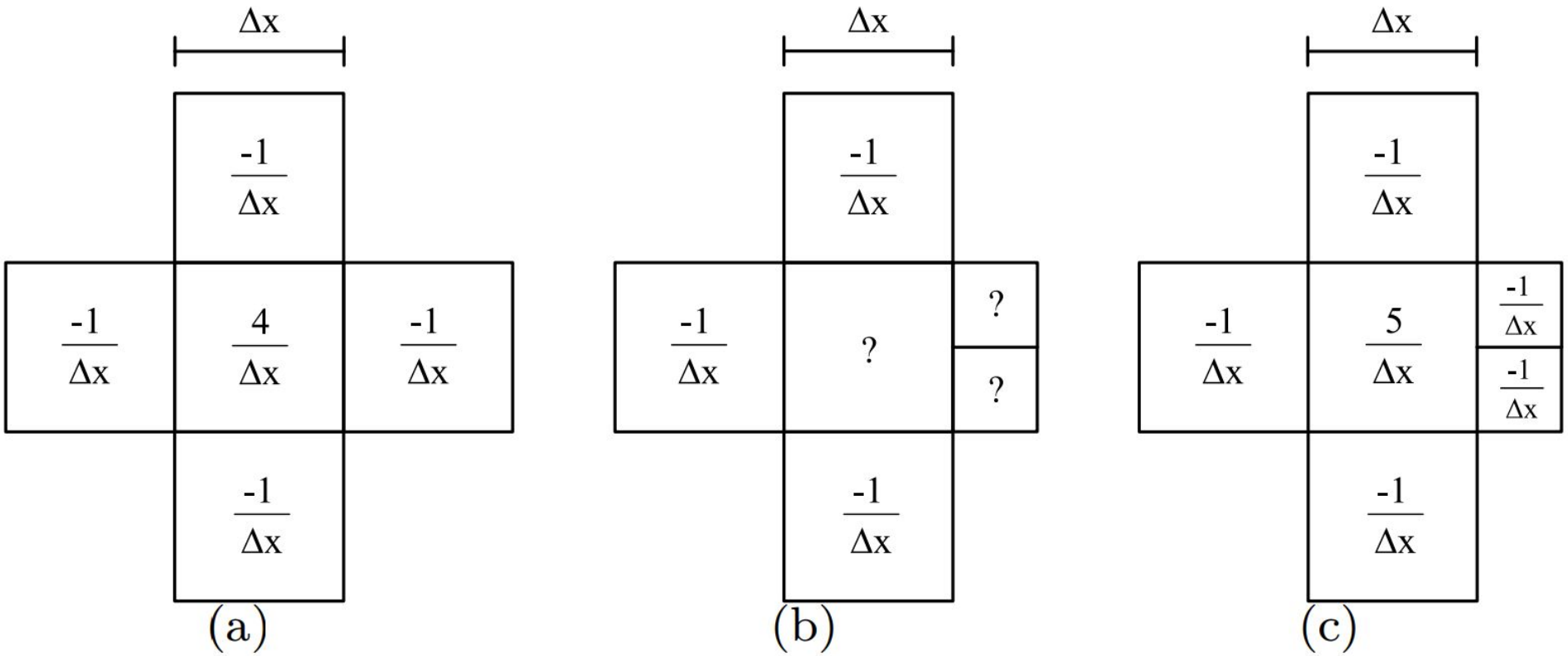



(a)



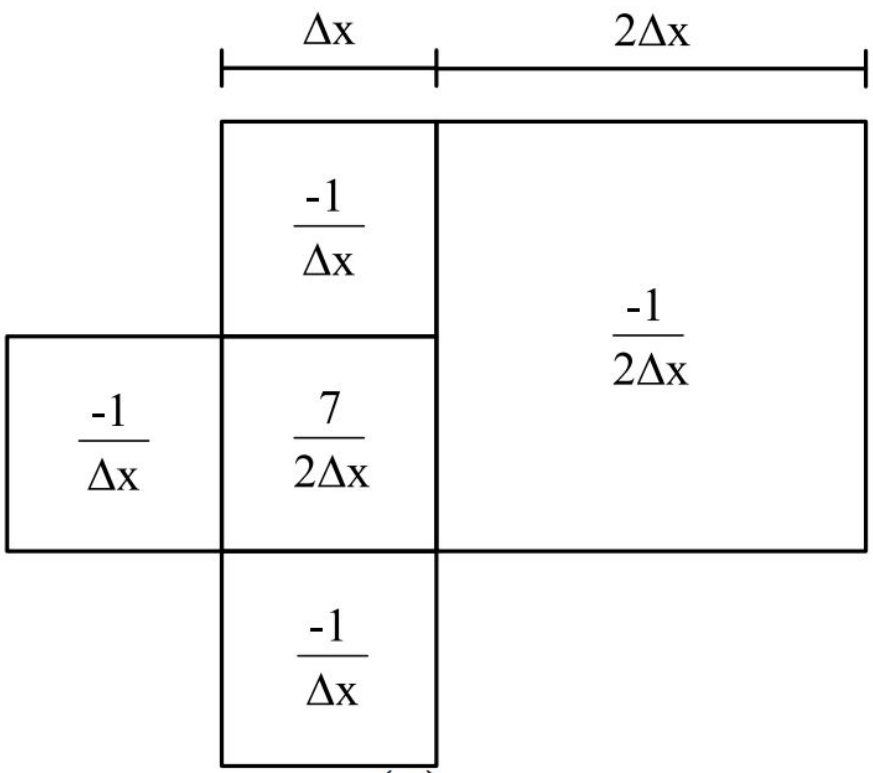

(b)

https://gamma.cs.unc.edu/FAST_LIGHTNING/lightning_tvcg_2007.pdf




https://gamma.cs.unc.edu/FAST_LIGHTNING/lightning_tvcg_2007.pdf

Delta notates the length, not Laplacian




(e)

https://gamma.cs.unc.edu/FAST_LIGHTNING/lightning_tvcg_2007.pdf

# Approximate it

- For CG purposes, approximation will do

# Fast Simulation of Laplacian Growth

Theodore Kim, Jason Sewall, Avneesh Sud and Ming C. Lin
University of North Carolina at Chapel Hill, USA
{kim,sewall,sud,lin}@cs.unc.edu

https://gamma.cs.unc.edu/FRAC/laplacian_small.pdf

# 3. Select growth locations based on the harmonic function

- Can be stochastic
- Single point of growth
- Surface growth

# 4. Repeat from step 2 until the growth is desirable

- Run a certain number of iterations
- Positive and negative boundaries touch
- Or meet some other criterium

# 5. Post-process

- Laplacian field alone is useless
- Turn it into a graph
- Volumetric rendering

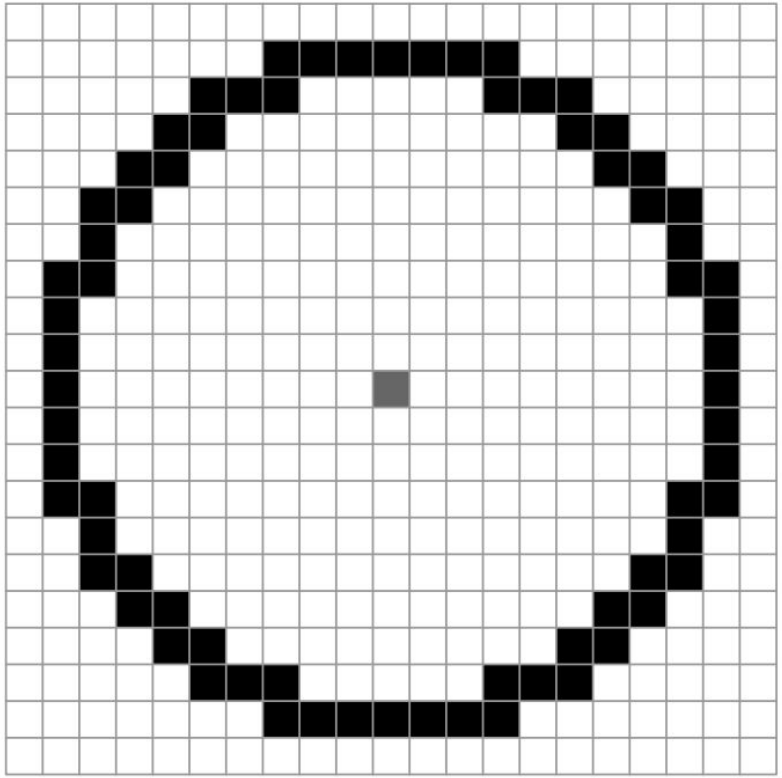
# Uses

https://gamma.cs.unc.edu/FRAC

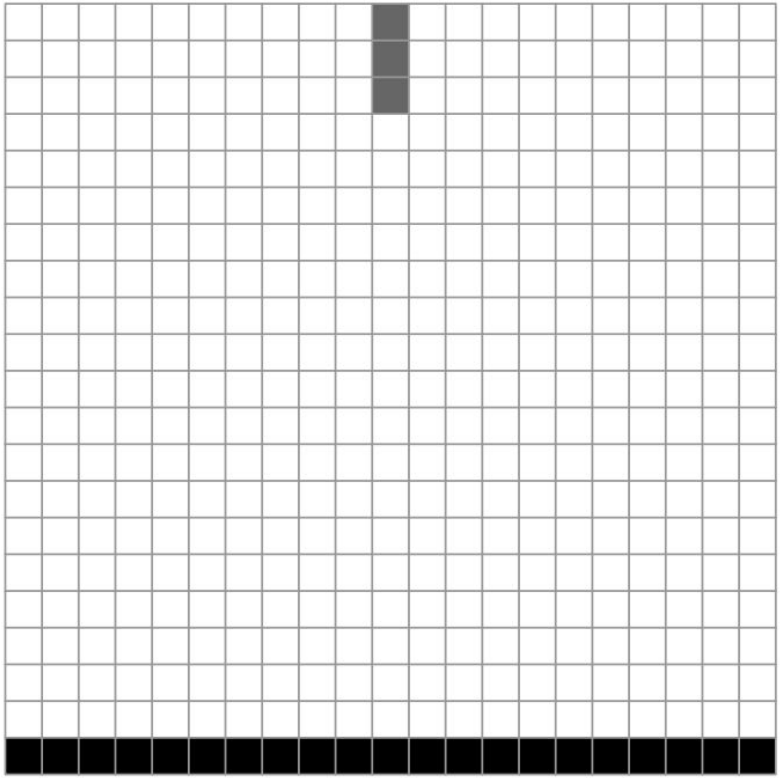
YouTube


FAR WAY

YouTube



(a) Original configuration



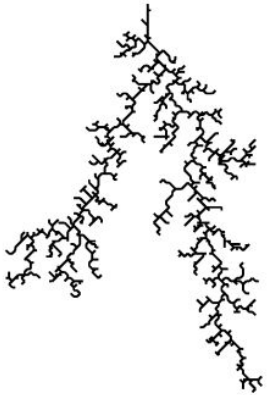
(b) Lightning configuration

https://gamma.cs.unc.edu/LIGHTNING/lightning.pdf

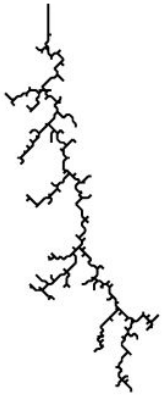
$$p_i = \frac{(\phi_i)^\eta}{\sum_{j=1}^{n} (\phi_j)^\eta}$$

https://gamma.cs.unc.edu/LIGHTNING/lightning.pdf

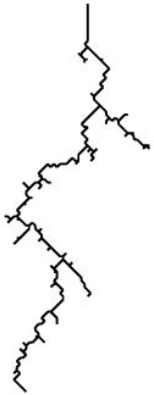
(a) Original configuration



(b) $\eta = 1$



(c) $\eta = 2$



(d) $\eta = 3$

https://gamma.cs.unc.edu/LIGHTNING/lightning.pdf

# Lightning is not just one bolt, but many subsequent very similar bolts

https://gamma.cs.unc.edu/LIGHTNING/lightning.pdf

Lightning is not just one bolt, but many subsequent very similar bolts

$$\left(\nabla^2 + \left(\frac{\omega}{c}\right)^2\right)\phi = -4\pi\rho$$

https://gamma.cs.unc.edu/LIGHTNING/lightning.pdf

Lightning is not just one bolt, but many subsequent very similar bolts

$$\left(\nabla^2 + \left(\frac{\omega}{c}\right)^2\right)\phi = -4\pi\rho$$

$$\nabla^2\phi = -4\pi\rho$$

https://gamma.cs.unc.edu/LIGHTNING/lightning.pdf

Rho is determined by previous bolts

# Post-processing

- Convert into a tree
- Determine main path
- Set the “wattage” of each edge

https://gamma.cs.unc.edu/LIGHTNING/lightning.pdf

## Post-processing

- Convert into a tree
- Determine main path
- Set the “wattage” of each edge

## Rendering

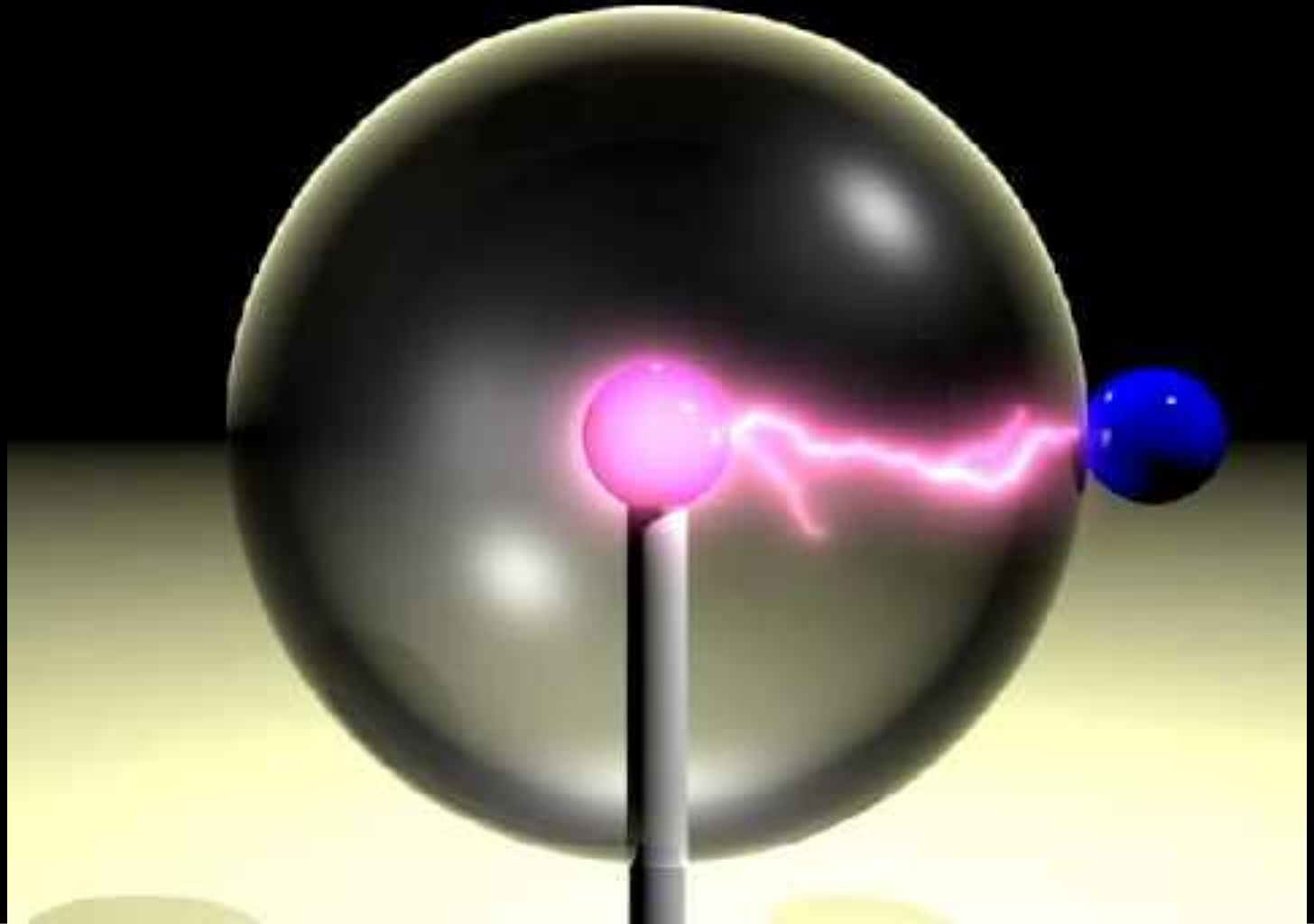
- Lightning bolt is actually very narrow
- Construct an APSF kernel
  - Authors created for 2km
- Use APSF kernel for post-processing

https://gamma.cs.unc.edu/LIGHTNING/lightning.pdf

ASPF - Atmospheric point spread function

https://gamma.cs.unc.edu/LIGHTNING/lightning.pdf


YouTube



YouTube

https://gamma.cs.unc.edu/FAST_LIGHTNING/lightning_tvcg_2007.pdf

# Potential uses

- Movies/animations
- Environments in games
- Gameplay

# Potential uses

- Movies/animations
- Environments in games
- Gameplay
  - Tesla coil ability in Sprash

https://courses.cs.ut.ee/2021/cg-pro/fall/Main/Project-Sprash

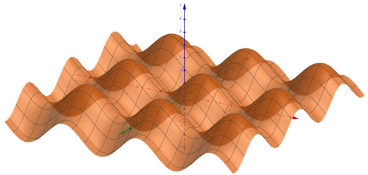
$$\nabla^2 \phi = 0$$

$$f(x, y) = e^x sin\, y$$

$$\Phi$$

# The End

$$p_i = \frac{(\phi_i)^\eta}{\sum_{j=1}^{n} (\phi_j)^\eta}$$



$$\nabla^2 \phi(v) = \sum_{w:d(w,v)=1} \phi(v) - \phi(w)$$

# Sources

Main sources:

- https://gamma.cs.unc.edu/LIGHTNING/
- https://gamma.cs.unc.edu/FRAC/laplacian_small.pdf
- https://gamma.cs.unc.edu/FAST_LIGHTNING/
- https://people.eecs.berkeley.edu/~demmel/cs267/

Wikipedia sources:

- https://en.wikipedia.org/wiki/Discrete_Laplace_operator
- https://en.wikipedia.org/wiki/Laplacian_matrix
- https://en.wikipedia.org/wiki/Laplace_operator
- https://en.wikipedia.org/wiki/Laplace%27s_equation

Used videos

- https://youtu.be/xkClgx88jR0
- https://youtu.be/x1piZoh_Nvk
- https://youtu.be/SL6SkFEZnf4
- https://youtu.be/HC0DFVRLV0U

Useful videos

- https://youtu.be/EW08rD-GFh0
- https://youtu.be/XbCvGRjjzgg
- https://youtu.be/JQSC0lCPG24

Pictures

- https://pixelero.wordpress.com/2011/11/18/laplacian-growth-vector-graphics/
- https://n-e-r-v-o-u-s.com/blog/?p=1536
- https://unsplash.com/photos/0G01UI1MQhg
- https://www.peridotfinejewelry.com/products/jamie-joseph-smooth-round-dendritic-agate-ring
- https://www.farwayart.com/shop/scenic-dendrite-agate
- https://stormandsky.com/lightning
- https://sites.und.edu/timothy.prescott/apex/web/apex.Ch15.S2.html
- https://favpng.com/png_view/sample-vector-curl-divergence-vector-field-vector-calculus-png/YwAaqMfb#_=_

Tools

- https://www.geogebra.org/3d/
- https://latexeditor.lagrida.com/
- MS Paint